PHILOSOPHICAL TRANSACTIONS,

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCVI.

PART I.

LONDON,

PRINTED BY W. BULMER AND CO. CLEVELAND-ROW, ST. JAMES'S;
AND SOLD BY G. AND W. NICOL, PALL-MALL, BOOKSELLERS TO HIS MAJESTY,
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ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the council-books and journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries, till the Forty-seventh Volume: the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to

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be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they receive them, are to be considered in no other light than as a matter of civility, in return for the respect shewn to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped, that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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Meteorological Journal kept at the Apartments of the Royal Society, by Order of the President and Council.

ERRATA IN THE MEMOIR ON IMAGINARY QUANTITIES.

Page 26, line 23, *for* 30. *read* 2de.

— 32, — 14. *for* $(1\pm\sqrt{1})$, *read* $(1\pm\sqrt{-1})$.

— 40, — 19. *for* $\sqrt{2}\times\sqrt{2}$, *read* $\sqrt{2}\times\sqrt{2}$.

— 69, — 15. *for* $+\frac{(0\times 2)^{90°}\sqrt{-2}}{1\cdot 2}$, *read* $+\frac{(0\times 2)^{90°}\sqrt{-1}^2}{1\cdot 2}$.

— 70, — 19, *for* cercle et, *read* centre à.

— 76, — 18, *for* $\left\{(\sin.\ a\sqrt{-1}\ \cos.\ a)^{\frac{1}{2}}\right.$, *read* $\left\{(\sin.\ a\sqrt{-1}+\cos.\ a)^{\frac{1}{2}}\right.$.

— 82, — 6, *for* $\pm px$, *read* $\pm\sqrt{px}$.

PHILOSOPHICAL TRANSACTIONS.

The PRESIDENT and COUNCIL of the ROYAL SOCIETY adjudged the Medal on Sir GODFREY COPLEY's Donation for the year 1805 to HUMPHRY DAVY, Esq. F. R. S. for his various communications published in the Philosophical Transactions.

PHILOSOPHICAL TRANSACTIONS.

I. *The Croonian Lecture on the Arrangement and mechanical Action of the Muscles of Fishes.* By Anthony Carlisle, *Esq. F.R.S. F.L.S.*

Read November 7, 1805.

It was my intention to have continued my physiological inquiries on the phenomena of muscular motion, by a series of chemical experiments; and to have communicated the result, when duly matured, to the Royal Society. But an unexpected request, made at a late period, for the Lecture of the present year, obliges me to defer those researches, and to limit the investigation of the subject I have chosen.

The application of the motive organs of animals has already furnished examples of general utility by increasing our knowledge of mechanical powers; and the cultivation of this study promises still further improvement.

The muscles of fishes are of a very different construction from those of the other natural classes. The medium in which these animals reside, the form of their bodies, and the

instruments employed for their progressive motion, give them a character peculiarly distinct from the rest of the creation. The frame-work of bones or cartilages, called the Skeleton, is simple; the limbs are not formed for complicated motions, and the proportion of muscular flesh is remarkably large. The muscles of fishes have no tendinous chords, their insertions being always fleshy. There are, however, semi-transparent, pearly tendons placed between the plates of muscles, which give origin to a series of short muscular fibres passing nearly at right angles between the surfaces of the adjoining plates. LEWENHOECK* appears to have overlooked these tendons, and the numerous vessels, which he describes in the interstices of the muscular flakes, I have not been able to discern.

The motion of a round shaped fish, independent of its fins, is simple; and as it is chiefly effected by the lateral flexure of the spine and tail, upon which the great mass of its muscular flesh is employed, whilst the fins are moved by small muscles, and those, from their position, comparatively but of little power, I shall only describe in detail the arrangement and application of those masses, which constitute the principal moving organs.

For this purpose a well known fish, the cod,† has been selected as a standard of comparison for the muscles of other fishes, there being a conspicuous resemblance among them all.

The pairs of fins have been considered as analogous to feet, but they are only employed for the purposes of turning, stopping, altering the position of the fish towards the horizon,

* Phil. Trans. Vol. XXXI. p. 190. † *Gadus Morhua* of LINNÆUS.

and for keeping the back upwards. The single fins appear to prevent the rolling of the body, whilst the tail is employed to impel it forward.

Each of those fins, which are in pairs, is capable of four motions, *viz.* of flexion and extension, like oars, and of expanding the rays, and closing them.

The extension of the whole fin is performed by a single radiated muscle, which is often supplied with red blood: the antagonist is of a similar character. The great power of the extensor muscle *(Vide* Plate I. *a, a,)* shews how strongly it is required to act when employed to stop suddenly the progressive motion. A series of intervening muscles expand and close the rays.

In the act of extending the fin the interosseal muscles are passive. It is advanced forward edgeways and closed; but during its flexion, the rays are expanded, striking the water with its broadest surface: this action assists the tail in turning the fish. In the effort to stop, these fins are strongly retained at right angles with the body, by the force of the extensor muscles, the rays are expanded, and the effect is assisted by the tail turning laterally with its broadest surface forward.

The single fins, for the expansion and contraction of their rays, are furnished with two sets of muscles; one of which is situated at their roots, and lies oblique; *(bbbbb)* the other, parallel with the spines, to which the rays are articulated *(cc.)* The fin has also a lateral motion, by which it is occasionally drawn out of a straight line; and by the co-operation of these muscles on both sides, it is kept steady whilst the body of the fish is turned oblique in swift motion, or in eddies.

When placed near the tail, the single fins seem also to aid the effect of that instrument by increasing its breadth.

The tail is the principal organ of progressive motion, and its actions are performed by the great mass of lateral muscles. There are a series of short muscles for the purpose of changing the figure of the tail fin, which arise from the spine and *coccyx*, and are attached to the rays immediately beyond their joints: *(dd)*: their action is to expand the rays, and by partial contractions to alter the lateral position of the fin. Slender muscles are placed between the several rays, *(ee,)* whose office is to converge them previous to the stroke of the tail.

The muscles situated on the head are those, which act on the *membrana branchiostega*, the under jaw, *os hyoides*, *fauces*, and the globe of the eye.

In order to determine the effect of the fins on the motions of fishes, a number of living dace,* of an equal size, were put into a large vessel of water. The pectoral fins of one of these fishes were cut off, and it was replaced with the others. Its progressive motion was not at all impeded; but, the head inclined downwards, and when it attempted to ascend, the effort was accomplished with difficulty.

The pectoral and abdominal fins were then removed from a second fish. It remained at the bottom of the vessel, and could not be made to ascend. Its progressive motion was not perceptibly more slow; but when the tail acted, the body shewed a tendency to roll, and the single fins were widely expanded, as if to counteract this effect.

From a third fish, the single fins were taken off. This

* *Cyprinus leuciscus.*

produced an evident tendency to turn round, and the pectoral fins were kept constantly extended to obviate that motion.

From a fourth fish, the pectoral and abdominal fins were cut off on one side, and it immediately lost the power of keeping the back upwards. The single fins were expanded, but the fish swam obliquely on its side with the remaining pectoral and abdominal fins downwards.

From a fifth fish, all the fins were removed. Its back was kept in a vertical position, whilst at rest, by the expansion of the tail, but it rolled half round at every attempt to move.

From a sixth fish, the tail was cut off close to the body. Its progressive motion was considerably impeded, and the flexions of the spine were much increased during the endeavour to advance: but neither the pectoral nor abdominal fins seemed to be more actively employed.

From a seventh fish, all the fins and the tail were removed. It remained almost without motion, floating near the surface of the water, with its belly upward.

These experiments were repeated on the roach,* the gudgeon,† and the minnow,‡ with similar results.

The muscles of fishes differ materially in their texture from those of other animals: they are apparently more homogeneous, their fibres are not so much fasciculated, but run more parallel to each other, and are always comparatively shorter. They become corrugated at the temperature of 136° of FAHRENHEIT, when their tendinous and ligamentous attachments are dissolved, and their serous juices coagulated. Under those circumstances the muscles lose their transparency, and the lateral cohesion of their fibres is lessened.

* *Cyprinus rutilus.* † *Cyprinus gobio.* ‡ *Cyprinus phoxinus.*

But the mechanical arrangement and physiology of the lateral muscles of the body of fishes constitute my present object. These parts have already been described in a general way by Professor CAMPER, M. VICQ-D-AZYR, and M. CUVIER, to whom I am indebted for much useful information. They have been denominated "*couches musculaires*" by M. VICQ-D-AZYR,* and "*muscles latérals*" by M. CUVIER.† The term used by M. CUVIER seems very appropriate for the general division or class. But, as the flakes are arranged in distinct longitudinal rows, these rows must be considered as orders. And, as "*couches*" appears objectionable, I shall adopt *series* in its stead; distinguishing each by a word referring to its situation in the animal, *viz.* the dorsal, vertebral, abdominal, and ventral series.

These series are composed of thin masses of muscle, or, as they are commonly called, flakes; which for the most part are thicker upon their outward edges, and become wedge-shaped towards their interior attachments. Each series is separated from the next adjoining by a membraneous partition, which is most apparent between the vertebral and abdominal series.

The dorsal series (*ff*) arises from the back of the head. In its course it is terminated on the upper edge by the bones, which support the single fins, and a membranous *septum*: at this part the flakes are thin. Its lower margin is bounded by the vertebral series, where the flakes become gradually thicker. The first flake is composed of longer fibres than the rest, and possesses more red blood. Those succeeding it

* *Mem. étrangers de l'Académ. des Sci. de Paris.* Tom. VII. p. 18. et 223.

† *Leçons d'Anatomie Comparée.* Vol. I. p. 196.

range obliquely backwards. They are all joined together by cellular membrane, and shining fasciæ, which resemble the tendinous expansions in quadrupeds.

Towards the middle of the fish the flakes are thicker, and stand more perpendicular to the surface, becoming oblique and thin as they approach the tail; whilst the intervening fasciæ are most dense at each extremity. This series consists of forty-five flakes, a number corresponding with that of the spinous processes to which they are attached, and which does not vary with the growth of the fish.

The muscular fibres constituting each flake, run nearly at right angles with its anterior and posterior surfaces, and parallel to the length and surface of the fish; except that their posterior extremities incline somewhat inwards.

As the skull affords the ultimate fixed attachment of this series, and its moveable insertions are on the vertebræ, and the tail, it follows, that its combined action is to bend the whole body and tail towards one side; or, if the flakes contract partially, to give it a serpentine motion. To produce these effects all the other series co-operate.

The superior external edges of the flakes of the vertebral series *(gg)* form acute angles with the inferior external edges of those of the dorsal series, the apices of which point towards the tail: the flakes are larger, but their number is the same. The lower margin of this series is bounded by the central membranous partition, which has already been noticed to be more conspicuous than the other longitudinal divisions, and it apparently admits of greater motion.

The abdominal series *(hh)* is composed of flakes similar to the preceding. They range towards the tail, forming an angle

with those of the vertebral series, the apex of which is presented towards the head. They are attached internally to the transverse and inferior spinous processes of the vertebræ. The ribs are placed in the line of the centre partition, and lie between the flakes. This series arises from a bone which borders the opening for the gills, and the pectoral fin, with its scapula and muscles, is situated between its foremost flakes. Wherever this series encloses the viscera, its flakes are shallow, and their thickness internally is not much less than at their external superficies.

Lastly, the flakes of the ventral series *(ii)* form acute angles with the abdominal flakes, the points of which incline to the tail. It is attached anteriorly to the *os hyoides*, and the bones of the lower jaw. In its course it is bounded above by the abdominal series, and below by a membranous *septum*, within which the inferior single fins arise. The flakes, that cover the viscera, are shallow; and they lie more oblique as they approach the tail. Both this, and the last described series, have their muscular fibres arranged according to the length and figure of the fish.

Three large superficial nerves *(kk)* passing longitudinally from the head to the tail, in the course of the membranous partitions, give off fibrils at right angles, which bend inwards between each of the muscular flakes. A larger set of nerves are sent from the *medulla spinalis*, one between each flake, the branches of which seem to enter without ramifying there. Another small nerve passing from the head, and running deep-seated, and close to the dorsal spines, crosses and unites with each of the spinal fibrils, and at the junction a remarkable body appears: it is a loose transparent vesicle, about the

size of a millet-seed, containing a white substance like the carbonate of lime found in the intercostal ganglions of frogs. This vesicle is included within the sheath of the nerve.

The coats of the blood-vessels are of a delicate texture, and easily ruptured. In order, therefore, to secure them from being injured by the violent and sudden actions of the muscles, the principal trunks both of the arteries and veins are inclosed in osseous canals, formed by the bases of the superior and inferior spinous processes; and their first ramifications lie within grooves in the spines. As they pass out to supply the muscles, their branches are immediately subdivided, so that a considerable vessel soon becomes extremely minute.

The rate, at which many fishes move through a medium so dense as water, is very remarkable; their velocity being scarcely surpassed by the flight of the swiftest birds: and although the large proportion of muscles, and their advantageous application, may partly account for the phenomenon, yet the power would be inadequate to the effect, if it were not suddenly enforced; as is evident from the slow progress of eels, and such fishes as are incapable from their length, and flexibility, of giving a sudden lateral stroke.

But the quickness and force of action in the muscles of fishes are counterpoised by the short duration of their powers. Those accustomed to the diversion of angling, are aware how speedily the strength of a fish is exhausted, for if, when hooked, it be kept in constant action, it soon loses even the ability to preserve its balance, and turns upon its side, fatigued and incapable of motion. This has been vulgarly attributed to drowning, in consequence of the mouth being closed upon

the hook; but the same effects take place when the hook is fastened to the side, or tail. This prostration of strength may depend partly on fear, and partly on interrupted respiration, since fishes, when swimming rapidly, keep the *membranæ branchiostegæ* closed, and when nearly exhausted, act violently with their gills.

The shortness of the muscular fibres, and the multiplied ramifications of the blood-vessels, are probably peculiar adaptations for the purpose of gaining velocity of action, which seems to be invariably connected with a very limited duration of it. Such examples form an obvious contrast with the muscular structure of slow-moving animals, and with those partial arrangements where unusual continuance of action is concomitant.

Since my former communications on the subject of cylindrical arteries,* another instance of their supplying slow-moving muscles, which are capable of long continued action, has been pointed out to me by Mr. MACARTNEY. It is in the muscles, which act upon the feet and toes of many birds, and seems to be an adaptation for the long exertion of those muscles while they sleep, and also when they alternately retract one foot under the feathers to preserve it from the effects of cold.

The muscles of the human body, which perform the most sudden actions, have their masses of fibres subdivided by transverse tendons, or are arranged in a penniform direction. The semi-tendinosus, and semi-membranosus of the thigh are thus constructed; the former having its fleshy belly divided by a narrow *fascia*, and the fibres of the latter being ranged

* Phil. Trans. 1800, p. 98.—Also 1804, p. 27.

in a half-penniform manner. The *recti abdominis* are also divided into short masses by transverse tendons, and all these muscles are conjointly employed in the action of leaping.

Perhaps these observations may indicate the reason for that diversity in the lengths of various muscles, which act together; thus, organs of velocity are joined with those of power, and mutually co-operate to produce a simultaneous effect.

Description of Plate I.

The drawing was made from a cod which had been coagulated by heat, in a case of plaister of Paris, the skin being taken away, and an equal portion of the flakes carefully removed from each series, to exhibit their several directions. The subject was reduced to the present size by accurate measurements.

aa, Muscles which extend the pectoral and jugular fins.

bbbbb, Oblique muscles, which erect the rays of the single fins.

cc, Muscles which depress the rays.

dd, Muscles which extend the rays of the tail.

ee, Interosseal muscles, which close the rays.

ff, The dorsal series of muscular flakes.

gg, The vertebral series.

hh, The abdominal series.

ii, The ventral series.

kkk, Three superficial nerves which run longitudinally between the series of flakes.

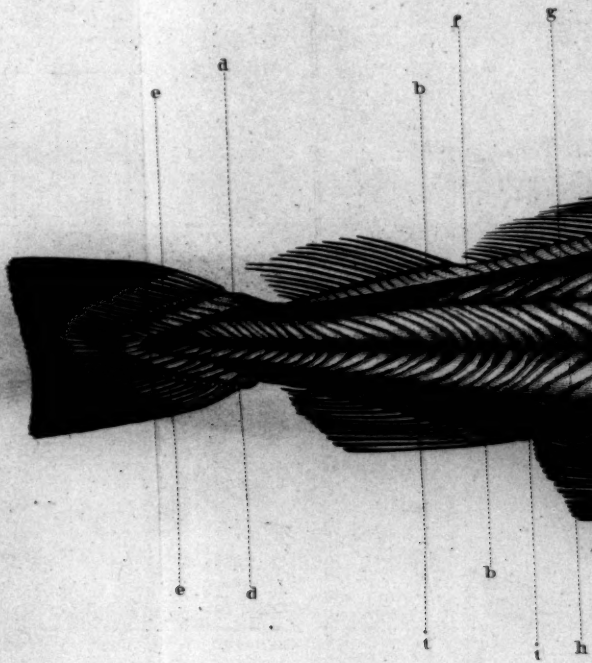
l, Posterior surface of a dorsal flake.

m, Posterior surface of an abdominal flake.

n, Anterior surface of a vertebral flake.

o, Anterior surface of an abdominal flake.

The middle portion of the fish from whence the flakes have been removed, shews the several directions of them, and also their different thicknesses. The spine appears in the chasm.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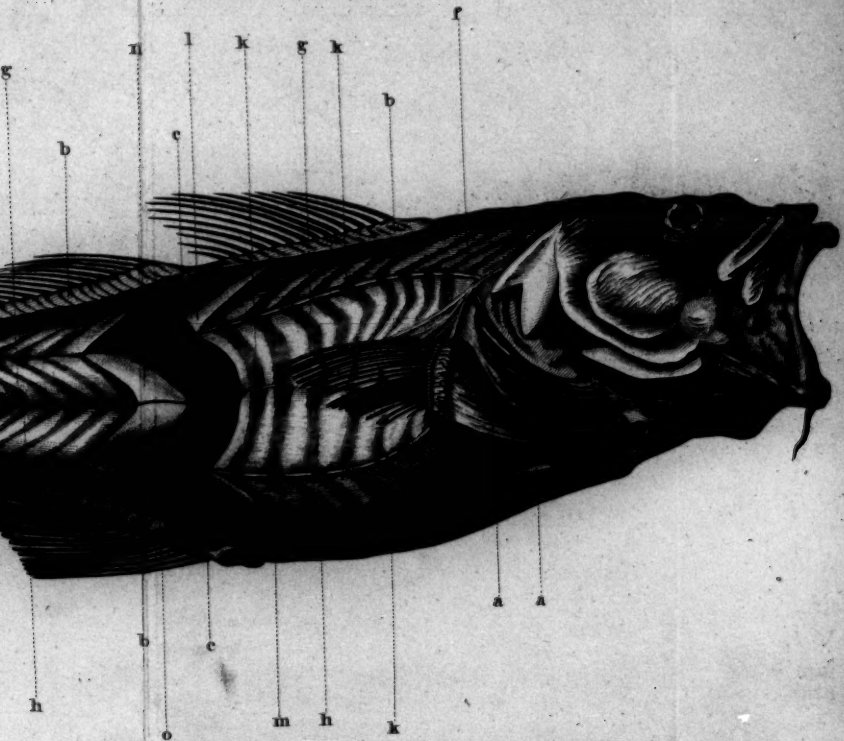

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II. *The Bakerian Lecture on the Force of Percussion.* By William Hyde Wollaston, *M. D. Sec. R. S.*

Read November 14, 1805.

When different bodies move with the same velocity, it is universally agreed that the forces, which they can exert against any obstacle opposed to them, are in proportion to the quantities of matter contained in the bodies respectively. But, when equal bodies move with unequal velocities, the estimation of their forces has been a subject of dispute between different classes of philosophers. Leibnitz and his followers have maintained that the forces of bodies are as the masses multiplied into the *squares* of their velocities, (a multiple to which I shall for conciseness give the name of *impetus*); while those, who are considered as Newtonians, conceive that the forces are in the *simple ratio* of the velocities, and consequently as the *momentum* or *quantitas motus*, a name given by Newton to the multiple of the velocity of a body simply taken into its quantity of matter.

It cannot be expected that at this time any new experiment should be thought of, by which the controversy can be decided, since the most simple experiments that have already been appealed to by either party have received different interpretations from their opponents, although the facts were admitted.

My object in the present Lecture is to consider which of these opinions respecting the force exerted by moving bodies is most conformable to the usual meaning of that word, and to shew that the explanation given by NEWTON of the third law of motion is in no respect favourable to those who in their view of this question have been called NEWTONIANS.

If bodies were made to act upon each other under the circumstances which I am about to describe, the leading phænomena would occur, which afford the grounds of reasoning on either side.

Let a ball of clay or of any other soft and wholly inelastic substance be suspended at rest, but free to move in any direction with the slightest impulse; and let there be two pegs similar and equal in every respect inserted slightly into its opposite sides. Let there be also two other bodies, A and B, of any magnitude, which are to each other in the proportion of 2 to 1; suspended in such a position, that when perfectly at rest they shall be in contact with the extremities of the opposite pegs without pressing against them. Now if these bodies were made to swing with motions so adapted that in falling from heights in the proportion of 1 to 4 they might strike at the same instant against the pegs opposite to them, the ball of clay would not be moved from its place to either side; nevertheless the peg impelled by the smaller body B, which has the double velocity, would be found to have penetrated twice as far as the peg impelled by A.

It is unnecessary to make the experiment precisely as here stated, since the results are admitted as facts by both parties; but upon these facts they reason differently.

One side observing that the ball of clay remains unmoved,

considers the proof indisputable that the action of the body A is equal to that of B, and that their forces are properly measured by their momenta, which are equal, because their velocities are in the simple inverse ratio of the bodies. Their opponents think it equally proved by the unequal depths to which the pegs have penetrated, that the causes of these effects are unequal, as they find to be the case in their estimation of the forces by the squares of the velocities.

One party is satisfied that equal *momenta* can resist equal pressures during the same *time*; the other party attend to the *spaces* through which the same moving force is exerted, and finding them in the proportion of 2 to 1, are convinced that the *vis viva* of a body in motion is justly estimated by its magnitude and the square of its velocity jointly.

The former conception of a quantity dependent on the continuance of a given *vis motrix* for a certain *time* may have its use, when correctly applied, in certain philosophical considerations; but the latter idea of a quantity resulting from the same force exerted through a determinate *space* is of greater practical utility, as it occurs daily in the usual occupations of men; since any quantity of work performed is always appreciated by the extent of effect resulting from their exertions: for it is well known, that the raising any great weight 40 feet would require 4 times as much labour as would be requisite to raise an equal weight to the height of 10 feet, and that in its slow descent the former would produce 4 times the effect of the latter in continuing the motion of any kind of machine. Moreover, if the weights so raised were suffered to fall freely through the heights that have been ascended by means of 4 and of 1 minute's labour, the velocities acquired

would be in the ratio of 2 to 1, and the squares of the velocities in proportion to the quantities of labour from which they originated, or as 4 to 1; and if the forces acquired by their descent were employed in driving piles, their more sudden effects produced would be found to be in that same ratio.

This species of force has been, first by BERNOUILLI and afterwards by SMEATON, very aptly denominated mechanic force; and when by force of percussion is meant the quantity of mechanic force possessed by a body in motion, to be estimated by its quantity of mechanic effect, I apprehend it cannot be controverted that it is in proportion to the magnitude of the body and to the square of its velocity jointly.

But of this quantity of force NEWTON no where treats, and has accordingly given no definition of it. If, after defining what he meant by the *quantitas acceleratrix*, and *quantitas motrix*, he had had occasion to convey an equally distinct idea of the *quantitas mechanica* resulting from the continued action of any force, he might, not improbably, have proceeded conformably to the definition given by SMEATON, and have added

——quantitas mechanica est mensura proportionalis spatio per quod data vis motrix exercetur;

or, if speaking with reference to the accumulated energy communicated to a body in motion,

——proportionalis quadrato velocitatis quam in dato corpore generat.

But, if we attend to the words of his preface to the first edition of his *Principia*, he evidently had no need of such a definition;

"Nos autem non artibus sed philosophiæ consulentes, deque "potentiis non manualibus sed naturalibus scribentes," &c.

And again, nearly to the same effect in the *Scholium*, which follows the laws of motion, "Cæterum mechanicam tractare "non est hujus instituti."

In the third law of motion he has on the contrary been supposed to speak of this force from an ambiguity in the signification of the words *actio* and *reactio*. By these, however, Newton certainly meant a mere *vis matrix* or pressure, as he himself explains them. "Quicquid premit vel trahit alterum, "tantundem ab eo premitur vel trahitur. Si quis lapidem "digito premit, premitur et hujus digitus a lapide," &c. The same meaning is equally evident from his demonstration of the third corollary to the laws, in which he asserts that the *quantitas motûs* of two or more bodies estimated in any given direction is not altered by their action upon each other. The demonstration begins thus:

"Etenim actio eique contraria reactio æquales sunt per legem "tertiam, ideoque per legem secundam æquales in motibus "efficient mutationes versus contrarias partes." Now, if he had considered the third law as implying equality of more than mere moving forces, there could have been no occasion to refer to the second law, with a view thence to deduce the equality of momenta produced.

Some authors however have interpreted the third law differently, and accordingly have expressed a difficulty in comprehending the simple illustration given by Newton. When they say that action is equal to reaction, they mean not only that the instantaneous intensity of the moving forces, or pressures opposed to each other, are necessarily equal, but

conceive also a species of accumulated force residing in a moving body, which is capable of resisting pressure during a *time* that is proportional to its momentum or quantitas motus. If it be of any real utility to give the name of force to this complex idea of *vis motrix* extended through time, as well as that of *momentum* to its effects when unresisted, it would be requisite to distinguish this force always by some such appellation as *momental* force; for it is to be apprehended that for want of this distinction many writers themselves, and it is certain that many readers of diquisitions on this subject have confounded and compared together *vis motrix*, *momentum*, and *vis mechanica*: quantities, that are all of them totally dissimilar, and bear no more comparison to each other, than lines to surfaces, or surfaces to solids.

In practical mechanics, however, it is at least very rarely that the *momentum* of bodies is in any degree an object of consideration: the strength of machinery being in every case to be adapted to the *quantitas motrix*, and the extent and value of the effect to be produced depending upon the *quantitas mechanica* of the force applied, or in other words to the space through which a given vis motrix is exerted.

The comparative velocities given by different quantities of mechanic force to bodies of equal or unequal magnitude have been so distinctly treated of by SMEATON, in a series of most direct experiments,* that it would be a needless waste of time to reconsider them in this place. So also, on the contrary, the quantities of extended mechanic effect producible by bodies moving with different quantities of impetus have been as clearly traced by the same accurate experimentalist.†

But there is one view, in which the comparative forces of

* Phil. Trans. Vol. LXVI. 450. † Vol. LXXII. 337.

impact of different bodies was not examined by SMEATON, and it may be worth while to shew that when the whole energy of a body A is employed without loss in giving velocity to a second body B, the *impetus* which B receives is in all cases equal to that of A, and the force transferred to B, or by it to any third body C, (if also communicated without loss, and duly estimated as a mechanic force,) is always equal to that from which it originated.

As the simplest case of entire transfer, the body A may be supposed to act upon B in a direct line through the medium of a light spring, so contrived that the spring is prevented by a ratchet from returning in the direction towards A, but expands again entirely in the direction towards B, and by that means exerts the whole force which had been wound up by the action of A, in giving motion to B alone. In this case, since the moving force of the spring is the same upon each of the bodies, the accelerating force acting upon B at each point is to the retarding force opposed to A at the corresponding points in the reciprocal ratio of the bodies, and the squares of the velocities produced and destroyed by its action through a given space will consequently be in that same ratio. The momentum, which is in the simple reciprocal ratio of the bodies, might consequently be increased at pleasure by the means proposed, in the subduplicate ratio of the bodies employed; and if momentum were an efficient force capable of reproducing itself, and of overcoming friction in proportion to its estimated magnitude, the additional force acquired by such a means of increase, might be employed for counteracting the usual resistances, and perpetual motion would be easily effected. But since the *impetus* remains unaltered, it is evident

that the utmost which the body B could effect in return would be the reproduction of A's velocity, and restitution of its entire mechanic force neither increased nor diminished, excepting by the necessary imperfection of machinery. The possibility of perpetual motion is consequently inconsistent with those principles which measure the quantity of force by the quantity of its extended effect, or by the square of the velocity which it can produce.

In estimating the utmost effect which one body can produce upon another at rest, the same result is obtained by employing *impetus* as ascensional force, according to HUYGENS; for if the body A were allowed to ascend to the height due to its velocity, and if by any simple mechanical contrivance of a lever or otherwise the body B were to be raised by the descent of A, it is well known that the heights of ascent would be reciprocally as the bodies; and consequently that the *square* of the velocity to be acquired by free descent of B would be in that ratio, and the quantity of mechanic force would be preserved as before unaltered.

It may be of use also to consider another application of the same energy, and to shew more generally that the same quantity of total effect would be the consequence not only of direct action of bodies upon each other, but also of their indirect action through the medium of any mechanical advantage or disadvantage; although the time of action might by that means be increased or decreased in any desired proportion. For instance, if the body supposed to be in motion were to act by means of a lever upon a spring placed at a certain distance from the centre of motion, the retarding force opposed to it would be inversely as the distance of the body

from the centre; and since the space through which the body would move to lose its whole velocity would be reciprocally as the retarding force, the angular motion of the lever and space through which the spring must bend, would be the same, at whatever point of the lever the body acted. And conversely, the reaction of the spring upon any other body B, would in all positions communicate to it the same velocity.

It may be remarked, however, that the times in which these total effects are produced may be varied at pleasure in proportion to the distances at which the bodies are placed from the centre of motion; and it should not pass unobserved that, although the intensity of any *vis motrix* is increased by being placed at what is called a mechanical advantage, yet on the contrary, any quantity of mechanic force is not liable to either increase or diminution by any such variation in the mode of its application.

Since we can by means of any mechanic force consisting of a *vis motrix* exerted through a given *space*, give motion to a body for the purpose of employing its *impetus* for the production of any sudden effect, or can, on the contrary, occasion a moving body to ascend, and thus resolve its *impetus* into a moving force ready to exert itself through a determinate space of descent, and capable of producing precisely the same quantity of mechanic effect as before, the force depending on *impetus* may justly be said to be of the same kind as any other mechanic force, and they may be strictly compared as to quantity.

In this manner we may even compare the force of a body in motion to the same kind of force contained in a given quantity of gunpowder, and may say that we have the same

quantity of mechanic force at command whether we have 1lb. of powder, which by its expansion could give to 1 ton weight a velocity sufficient to raise it through 40 feet, or the weight actually raised to that height and ready to be let down gradually, or the same weight possessing its original velocity to be employed in any sudden exertion.

By making use of the same measure as in the former cases, a distinct expression is likewise obtained for the quantity of mechanic force given to a steam-engine by any quantity of coals; and we are enabled to make a comparison of its effect with the quantity of work that one or more horses may have performed in a day, each being expressed by the space through which a given moving force is exerted. In the case of animal exertion however, considerable uncertainty always prevails in consequence of the unequal powers of animals of the same species, and varying vigour of the same animal. The information which I have received in reply to inquiries respecting the weights raised in one hour by horses in different situations, has varied as far as from 6 to 15 tons to the height of 100 feet. But although the rate at which mechanic force is generated may vary, any quantity of work executed is the same, in whatever time it may have been performed.

In short, whether we are considering the sources of extended exertion or of accumulated energy, whether we compare the accumulated forces themselves by their gradual or by their sudden effects, the idea of mechanic force in practice is always the same, and is proportional to the *space* through which any moving force is exerted or overcome, or to the *square* of the velocity of a body in which such force is accumulated.

III. *Mémoire sur les Quantités imaginaires. Par M.* Buée. *Communicated by* William Morgan, *Esq. F. R. S.*

Read June 20, 1805.

Des Signes + et —

1. Ces signes ont des significations opposées.

Considérés comme signes *d'opérations arithmétiques*, + et — sont les signes, l'un de l'*addition*, l'autre de la *soustraction*.

Considérés comme signes d'*opérations géométriques*, ils indiquent des *directions opposées*. Si l'un, par exemple, signifie qu'une ligne doit être tirée de gauche à droite, l'autre signifie qu'elle doit être tirée de droite à gauche.

2. Remarque. Lorsqu'on décrit une ligne d'une longueur déterminée, dans une direction déterminée, on fait deux choses: 1°. on donne à cette ligne sa *longueur*; 2°. on lui donne sa *direction*. La première de ces opérations est purement *arithmétique*. La seconde est purement *géométrique*. Dans la première on fait abstraction de la direction. Dans la seconde on fait abstraction de la longueur. Lors donc qu'on réunit ces deux opérations, on fait réellement une opération *arithmético-géométrique*. Ainsi, lorsque je parlerai d'*opérations géométriques*, je n'aurai en vue que les *directions* des lignes, abstraction faite de leurs longueurs. Revenons aux signes + et —.

3. En général lorsque + et — ne signifient pas simple-

ment, l'un l'addition, et l'autre la soustraction, pour savoir ce que — signifie devant une lettre, il faut savoir ce que signifieroit + devant cette même lettre, et prendre pour — la signification opposée.

Si, par exemple, $+t$ signifie un *temps passé*, $-t$ signifie un *temps futur* égal. Si $+p$ désigne une *propriété*, $-p$ désigne une *dette* de même valeur, &c.

4. Il est important de remarquer ici que, lorsqu'une chose désignée par $+l$, une ligne, par exemple, change de situation dans le courant d'une opération arithmético-géométrique, et qu'en conséquence cette ligne a successivement plusieurs situations (qui toutes sont désignées par $+l$) il ne suffit pas, pour connoître la situation désignée par $-l$, de connoître une de celles qu'on désigne par $+l$; il faut encore savoir à laquelle chaque $-l$ est opposée.

5. Ce détail nous mène à la conséquence suivante.

Chacun des signes + et — a deux significations tout-à-fait différentes.

1°. Mis devant une quantité q, ils peuvent désigner, comme je l'ai dit, deux *opérations arithmétiques* opposées dont cette quantité est le sujet.

2°. Devant cette même quantité, ils peuvent désigner deux *qualités* opposées ayant pour sujet les *unités* dont cette quantité est composée.

6. Dans l'algèbre ordinaire, c'est-à-dire, dans l'algèbre considérée comme *arithmétique universelle*, où l'on fait abstraction de toute espèce de qualité, les signes + et — ne peuvent avoir que la première de ces significations.

Par conséquent, dans cette algèbre où tout est abstrait, une quantité isolée peut bien porter le signe + qui, dans ce cas,

n'ajoute rien à l'idée de cette quantité; mais elle ne peut pas porter le signe —. (C'est un des principes fondamentaux de Mr. Carnot, dans son excellent traité intitulé: *Géométrie de Position*.) En effet cette quantité étant supposée isolée, si on l'ajoute, ce ne peut être qu'à zéro; si on la soustrait, ce ne peut être que de zéro. Le premier est possible; mais le second est absurde. C'est aussi ce que Mr. Frend a bien vu, dans ses élémens d'algèbre.

Par conséquent, toutes les fois qu'on a pour résultat d'une opération une quantité précédée du signe —, il faut, pour que ce résultat ait un sens, y considérer quelque qualité. Alors l'algèbre ne doit plus être regardée simplement comme une *arithmétique universelle*, mais comme une *langue mathématique*.

7. Première remarque. Lorsqu'on a dit* que les signes + et — indiquoient deux sens opposés, on avoit en vue la seconde des significations que j'ai données à ces signes (No. 5); car être susceptible d'un sens est une qualité.

De même, lorsqu'on a dit† qu'une quantité négative étoit plus petite que zéro, on avoit encore en vue cette seconde signification; car ce n'est pas la *quantité* qui est plus petite que *zéro*; c'est la *qualité* qui est inférieure à la *nullité*. Par exemple, si mes dettes excèdent mes propriétés, je suis plus pauvre que si je n'avois ni propriétés ni dettes.

8. Seconde remarque. Les deux significations de + et — ne peuvent pas avoir lieu en même temps, relativement au même + ou au même —; car ce seroit faire signifier en même temps, au même signe, une *abstraction* et une *non-abstraction* de toute *qualité*. Mais 1°. ces deux significations

* *Voyez les Opuscules mathématiques de* D'Alembert, Tome VIII. p. 270.

† Euler, *Introductio in Analysin Infinitorum*, T. II. p. 4, No. 3.

peuvent avoir lieu *en même* temps pour deux + ou deux − différens. 2°. Elles peuvent aussi avoir lieu pour le même + ou le même −, *en deux différens temps*.

Le premier cas arrive dans ce qu'on appelle *la multiplication des signes*. Le second arrive dans la solution d'un problème. On sait en effet que, pour résoudre algébriquement un problème, il faut d'abord traduire la question en langage algébrique; ensuite traduire les formules du langage algébrique en d'autres formules du même langage; enfin traduire celles-ci dans les opérations qu'elles signifient. Lorsqu'on traduit une question (dont l'objet n'est pas quelque nombre abstrait) en langue algébrique, c'est la seconde signification (No. 5) qu'on doit attribuer aux signes + ou −. Dans la seconde traduction, c'est la première signification. Dans la 3e. c'est la première ou la 2de. ou toutes les deux à la fois (No. 2).

Il arrive dans toutes les solutions de problèmes par l'algèbre ce qui arrive dans les plus simples règles de trois. Lorsqu'on traduit la condition de la question en une proportion géométrique, on pose les termes de cette proportion comme des nombres *concrets*, c'est-à-dire, comme des nombres d'unités auxquelles des qualités sont attachées. Lorsqu'on opère une multiplication et une division sur les termes de cette proportion (c'est-à-dire, lorsqu'on fait la 3e. des traductions dont j'ai parlé) on regarde ces termes comme des nombres *abstraits*. Lorsqu'enfin on traduit en langue vulgaire le résultat de ces opérations, on regarde ce resultat comme un nombre *concret*.

9. Troisième remarque. Selon la seconde signification donnée (No. 5) aux signes + et −, ils désignent deux *qualités* opposées ayant pour sujets les *unités* dont une quantité

est composée. Or comme une qualité ne peut être séparée de son sujet, les signes + et − ne peuvent être séparés de leurs unités. Dans la langue algébrique, ces unités sont des *substantifs*, et les signes + et −, des *adjectifs*. Par conséquent $+q$ et $-q$ tiennent toujours lieu de $+1.q$ et $-1.q$, c'est-à-dire, de l'unité $+1$ ou -1 (ayant une qualité quelconque) prise q fois. Cette expression (q fois) marque que q est pris pour un nombre abstrait. De même, si l et l' désignent des lignes, $+l \times l'$ et $-l \times l'$ tiennent lieu de $+1^{s}.l \times l'$ et $-1^{s}.l \times l'$, 1^{s} etant une surface carrée et $l \times l'$ un nombre abstrait. Si l ou l' désignoit une surface, alors $+l \times l'$ et $-l \times l'$ (qui auroient chacun trois dimensions, savoir, les deux dimensions de la surface et la dimension de la ligne) tiendroient lieu de $+1^{s} \times ll'$ et $-1^{s} \times ll'$.

Il en seroit de même de toute autre signification de l et de l'. On voit par là toute l'étendue de la signification des adjectifs + et − unis à leur substantif 1.

Du Signe $\sqrt{-1}$.

10. Je mets en titre, *Du signe* $\sqrt{-1}$, et non, *De la quantité ou De l'unité imaginaire* $\sqrt{-1}$; parceque $\sqrt{-1}$ est un signe particulier joint à l'unité réelle 1, et non une quantité particulière. C'est un nouvel adjectif joint au substantif ordinaire 1, et non un nouveau substantif.

Mais que veut dire ce signe? Il n'indique ni une addition, ni une soustraction, ni une suppression, ni une opposition par rapport aux signes + et −. Une quantité accompagnée de $\sqrt{-1}$ n'est ni additive, ni soustractive, ni égale à zéro. La

qualité marquée par $\sqrt{-1}$ n'est opposée ni à celle qu'indique $+$, ni à celle qui est désignée par $-$. Qu'est-elle donc?

Pour le découvrir, supposons trois lignes égales AB, AC, AD, (Plate II. Fig. 1.) qui partent toutes du point A. Si je désigne la ligne AB par $+1$, la ligne AC sera -1, et la ligne AD, qui est une moyenne proportionelle entre AB et AC, sera nécessairement $\sqrt{-1^2}$, ou plus simplement, $\sqrt{-1}$. Ainsi $\sqrt{-1}$ est le signe de la PERPENDICULARITE', dont la propriété caractéristique est, *que tous les points de la perpendiculaire sont également éloignés de points placés à egales distances, de part et d'autre de son pié.* Le signe $\sqrt{-1}$ exprime tout cela, et il est le seul qui l'exprime.

Ce signe mis devant a (*a* signifiant une ligne ou une surface) *veut donc dire: qu'il faut donner à* a *une situation perpendiculaire à celle qu'on lui donneroit, si l'on avoit simplement* $+$ a *ou* $-$ a.

11. Voici une autre manière de parvenir au même résultat.

Soient AB, AD (Fig. 2.) deux côtés contigus du carré ABCD. Supposons $AB = \pm 1$, et par conséquent $AD = \pm 1$, et mettons en A le point de départ de la description des lignes AB et AD, ensorte que AB et AD portent le même signe $+$ ou $-$, et que le carré ABCD soit positif.

Maintenant faisons faire à ce carré ABCD un quart de révolution autour du point A pris comme centre. Après ce mouvement, le point B sera en B', le point C en C', et le point D en D'. Chacune des lignes AB, BC, CD, DA, prendra une situation perpendiculaire à celle qu'elle avoit, et, au lieu du carré ABCD, on aura le carré AB'C'D'. Or A étant le point de départ, il est clair que, si le carré ABCD est positif, le carré AB'C'D' doit être négatif, et *vice versâ*. Par

conséquent si $ABCD = +1$, dont le côté AB ou BC ou CD ou DA est $= \pm 1$, on a $AB'C'D' = -1$, dont le côté AB' perpendiculaire à AB, ou $B'C'$ perpendiculaire à BC, ou $C'D'$ perpendiculaire à CD, ou $D'A$ perpendiculaire à DA est $= \pm \sqrt{-1}$. On voit donc que, si l'on donne à tous les côtés d'un carré des positions perpendiculaires à celles qu'ils ont, *sans cependant changer leurs positions respectives et en faisant le plus petit mouvement possible* (c'est-à-dire, en n'ajoutant pas le mouvement de translation à celui de rotation) on obtient le même résultat qu'en joignant le signe $\sqrt{-1}$ au signe de ces côtés.

12. $\sqrt{-1}$ n'est donc pas le signe d'une opération *arithmétique*, ni d'une opération *arithmético-géométrique* (No. 9), mais d'une opération purement géométrique. C'est un signe de perpendicularité. C'est un signe purement descriptif. J'appelle signe *purement descriptif* un signe qui indique la *direction* d'une ligne, abstraction faite de sa *longueur*. Ainsi les mots *purement descriptif* ont la même signification que les mots *purement géométrique* (No. 9).

13. Il faut distinguer la perpendicularité indiquée par ce signe de celles qu'indiquent les signes *sin.* et *cos.* Ces derniers signes ne peuvent pas indiquer la perpendicularité l'un sans l'autre, et même si l'un et l'autre ne sont pas attachés à la même quantité. Ainsi *sin. a* et *cos. a* indiquent bien la perpendicularité de l'un à l'autre; mais *sin. a* et *cos. b* ne l'indiquent pas. $a\sqrt{-1}$, au contraire, indique relativement à a une situation perpendiculaire à celles de $+a$ et de $-a$.

Sin. et cos. sont des signes artificiels. $\sqrt{-1}$ est un signe

naturel, puisqu'il est une conséquence nécessaire des signes + et — considérés comme signes de direction.

14. La *perpendicularité* indiquée par le signe $\sqrt{-1}$ est une *qualité*. Par conséquent une quantité accompagnée de ce signe n'est pas une quantité abstraite, parceque ses unités ne sont pas des unités abstraites.

15. Non seulement l'unité affectée du signe $\sqrt{-1}$ n'est pas une unité abstraite, mais elle peut être regardée comme une nouvelle indéterminée introduite par ce signe. En effet, ce signe indique la perpendicularité, mais il n'indique que cela. Il n'indique pas le point de départ de la perpendiculaire. Si donc ce point de départ n'est pas déterminé d'ailleurs, ce signe le laisse indéterminé. Ainsi tandis que la *longueur* de la ligne perpendiculaire est *constante*, sa *manière d'être perpendiculaire* est *variable*. De plus, soit AD (Fig. 1.) l'unité à laquelle le signe $\sqrt{-1}$ est attaché. Elle peut être aussi bien AD', AD'', ou tout autre rayon du cercle DD'D''; puisque ce cercle etant supposé perpendiculaire au plan de ce papier, tous ses rayons sont perpendiculaires à la ligne CB qui est sur le plan de ce papier.

16. Quoique la *perpendicularité* soit *proprement* la seule qualité indiquée par le signe $\sqrt{-1}$, on peut lui faire signifier, *au figuré*, une qualité toute différente, pourvû qu'on puisse raisonner sur cette qualité, comme on raisonneroit sur la perpendicularité même. Par exemple, si $+s$ représente une somme *possédée*, et $-s$ la même somme *due*, je dis que $s\sqrt{-1}$ peut représenter la même somme *ni possédée ni due*, parcequ'on peut raisonner sur cette dernière somme relativement

aux deux autres, comme sur la ligne AD (Fig. 1.) relativement aux lignes AB, AC.

En effet, de même qu'un point quelconque de la ligne AD est également distant des points de la ligne CD qui se trouvent au même éloignement du point A, de même une partie quelconque de la somme qui n'est *ni possedée ni due* est dans une égale situation relativement aux parties égales de la somme *possédée* et de la somme *due*. La *possession active* étant donc exprimée par $+$, et la *dette ou possession passive*, par $-$, la *négation*, non pas de la somme, mais de sa *possession soit active soit passive*, peut toujours être exprimée par $\sqrt{-1}$.

17. On peut, d'après cette idée, résoudre facilement la question suivante.

Problème I.

Un homme possède un nombre n de livres. Il y a de plus un nombre n' de livres qui est par rapport à lui, ou une *propriété*, ou une *dette*, ou une *somme ni possedée ni due*. Si l'on ajoute ensemble les deux nombres n et n', on aura une somme x. Si l'on soustrait n' de n, on aura un différence y. Cette somme et cette différence sont telles que $x+y=a$, et $xy=b$ (b étant une somme *possedée* ou *due* ou *ni l'un ni l'autre*, et a une *propriété*). On demande 1°. les valeurs des nombres n et n'; 2°. si n' est *une propriété ou une dette ou une somme ni possédée ni due*?

D'après l'énoncé de cette question, nous avons les quatre équations suivantes: $x+y=a$; $xy=b$; $n+n'=x$; $n-n'=y$. Ces quatre équations donnent:

$$x=\frac{a\pm\sqrt{a^2-4b}}{2}; \quad y=\frac{a\mp\sqrt{a^2-4b}}{2}; \quad n=\frac{a}{2}; \quad \text{et } n'=\frac{\pm\sqrt{a^2-4b}}{2}.$$

Si $4b > a^2$, b étant positif, n' est une somme *ni possédée ni due*.

Si $4b < a^2$, ou si b est négatif, n' est une somme *ou possédée ou due*. Dans ce cas elle augmente ou diminue la somme désignée par n.

Si b porte le signe $\sqrt{-1}$, alors n' est en partie *possedée ou due* et en partie *ni possedée ni due*.

18. Il se présente ici une objection qui d'abord paroît insurmontable. Pour exposer cette objection de la manière la plus simple, substituons des chiffres aux lettres a et b. Faisons $a=2$ et $b=2$. Nous aurons $x=1\pm\sqrt{-1}$; $y=1\mp\sqrt{-1}$; $n=1$; et $n'=\sqrt{-1}=$ la valeur 1 qui n'est ni possedée ni due; $x+y=\quad 1\pm\sqrt{-1}\quad+1\mp\sqrt{-1}\quad=2$; $xy=(1\pm\sqrt{1})\ (1\mp\sqrt{-1})=1^2-(\sqrt{-1})^2=1^2-(-1)=2$. Ce resultat répond à la question; mais on peut croire qu'il n'y répond que d'après une supposition absurde, savoir, que $\sqrt{-1}$, qui n'est ni une propriété ni une dette, étant multiplié par $-\sqrt{-1}$, qui n'est pareillement ni une propriété ni une dette, donne $+1$ qui est une propriété. Comment suffit-il, pour me faire acquérir une propriété, de multiplier une somme qui m'est étrangère par une somme qui m'est également étrangère?

Cette objection est semblable à celle-ci: comment la dette -1 multipliée par la dette -1 peut-elle produire la propriété $+1$? Ce qui fait la force de cette objection tient à ce que, en la proposant, on n'analyse pas exactement ce qui se passe dans l'opération appellée *multiplication*. En effet, si l'on attache aux mots leur signification ordinaire, cette expression, *multiplier une dette par une dette*, ne présente aucun sens

intelligible; car, multiplier, par exemple, 8 par 4, signifie en général, prendre 8, 4 *fois*; de sorte que *multiplier la dette* 8 *par la dette* 4 ne pourroit signifier que, prendre la *dette* 8 *une dette* 4 *de fois*, ce qui est un galimathias. Cette expression -1×-1 ne doit donc pas être traduite par celle-ci: *la dette* -1 *multipliée par la dette* -1. Voyons comment elle doit être traduite.

Le multiplicateur -1 présente deux idées, savoir, l'idée de l'unité 1 et l'idée exprimée par le signe $-$. Ce dernier signe représente une opération, de sorte que le double signe $\times -$ en représente deux. Cela posé, il est facile de concevoir que -3×-4 signifie: -3 *pris* 4 *fois avec un signe contraire* à celui que donneroit $-3 \times +4$ ou -3.4 (No. 3.) Les mots: *pris* 4 *fois*, sont la traduction de $\times 4$, et les mots: *avec un signe contraire*, &c. sont celle de $-$. Maintenant si l'on applique ce qui vient d'être dit au quadruple signe $\sqrt{-1} \times -\sqrt{-1}$, on verra que les mots suivans de l'objection: $\sqrt{-1}$, *qui n'est ni une propriété ni une dette, étant multiplié par* $-\sqrt{-1}$, *qui n'est pareillement ni une propriété ni une dette*, que ces mots, dis-je, sont une fausse traduction de $\sqrt{-1} \times -\sqrt{-1}$, et que la vraie traduction de ce signe composé est celle-ci: *La quantité concrète* $\sqrt{-1}$ *prise une fois, dans un sens également éloigné des sens que présentoient* $\sqrt{-1} \times -(+1)$ et $\sqrt{-1} \times -(-1)$ (Nos. 10 et 16).

Cette explication fait évanouir l'objection. En effet cette objection tombe sur la manière dont la solution du problème remplit la condition exprimée par $xy = b$. xy signifie x *multiplié par* y. Cette expression: *multiplié par* y, doit être prise dans un sens intelligible. Or l'explication que je viens de

donner du signe complexe $\sqrt{-1} \times -\sqrt{-1}$ est (si je ne me trompe) parfaitement intelligible ; elle est même la seule qui puisse l'être ; elle n'offre d'ailleurs aucune absurdité, et conduit à la solution complette du problème proposé. Donc &c.

19. La solution de la question précédente me paroît impossible par l'algèbre ordinaire qui n'attribue aucune signification intelligible au signe $\sqrt{-1}$.

20. On voit par le détail dans lequel je viens d'entrer que, de cela seul qu'une question conduit à un résultat qui renferme le signe $\sqrt{-1}$, il ne s'ensuit pas qu'elle soit absurde.

Il ne s'ensuit pas non plus qu'elle ne le soit pas.

Pour aider à reconnoître ce qui en est, dans les différens cas, posons quelques principes.

21. 1°. Dans les questions purement arithmétiques, le signe $\sqrt{-1}$ devant indiquer une opération arithmétique, et ne le pouvant pas, indique une opération absurde.

22. 2°. $\sqrt{-1}$ indique une *qualité moyenne* entre deux qualités opposées dont l'une est exprimée par $+$ et l'autre par $-$. *Ces qualités doivent être indépendantes de la quantité.* Lors donc que la qualité moyenne n'est pas indépendante de la quantité, elle ne peut être exprimée par $\sqrt{-1}$; car $\sqrt{-1}$ ne peut exprimer qu'une qualité indépendante de toute quantité, c'est-à-dire, une qualité qui reste la même, quoique la quantité varie. Si, par exemple, j'exprime un temps *futur* par $+t$, et un temps *passé* par $-t$, $t\sqrt{-1}$ ne peut rien signifier, parceque le *présent* qui est la qualité moyenne entre le *futur* et le *passé*, n'est qu'un instant indivisible et qu'il n'a d'autre expression que o.

23. Si cependant on entend (car le langage ordinaire est

si vague qu'il est peu d'expressions qu'on ne puisse entendre de plusieurs manières) par *present*, un certain espace de temps, comme, *ce jour-ci*, *le mois présent*, *la présente année*, *le siècle present*, alors $t\sqrt{-1}$ peut avoir une signification. Car, par exemple, soit AB (Fig. 31) le mois passé $-t$, EF le mois prochain $+t$, BC représentera la première moitié $\frac{-t\sqrt{-1}}{2}$ et DE la seconde moitié $\frac{+t\sqrt{-1}}{2}$ du mois présent, de sorte que l'expression du mois présent entier sera $\frac{-t\sqrt{-1}}{2}+\frac{t\sqrt{-1}}{2}=0$, (voyez le No. suivant). Or 0 (qui, comme on le verra bientôt, a deux significations) est la véritable expression du présent. De plus, comme les lignes BC, DE, sont égales et perpendiculaires aux lignes AB, EF, on voit que les parties des deux moitiés du mois *présent* qui sont également éloignées de son milieu CD, le sont pareillement, l'une de l'extrémité A du mois *passé*, et l'autre de l'extrémité F du mois *prochain*.

24. Il faut cependant observer que, cette espèce de *présent* ayant des limites constantes, si la valeur de $\pm t\sqrt{-1}$ donnoit des limites différentes de celles qui seroient supposées tacitement, alors il y auroit une contradiction entre cette valeur de $\pm\sqrt{-1}$ et cette supposition tacite. Par conséquent la solution donnée par cette valeur seroit absurde.

25. N°. On trouvera peut-être une espèce de paralogisme dans l'équation $-\frac{t\sqrt{-1}}{2}+\frac{t\sqrt{-1}}{2}=0$, par laquelle je fais l'espace d'un mois égal à zéro. Mais on observera 1°. que cette équation ressemble à la phrase d'un homme qui, après s'être égaré se retrouve au point dont il vouloit s'éloigner, et dit: "Je ne suis pas plus avancé, après tant de chemin, que si je n'en avois fait aucun;" car le *temps* est pour l'esprit ce

que l'*espace* est pour le corps. On observera 2°. que $\frac{-i\sqrt{-1}}{2} + \frac{i\sqrt{-1}}{2}$ n'est qu'un signe, aussi bien que o. Ce ne sont pas les choses que j'égalise, mais les signes qui présentent ces choses sous un point de vue particulier. Je les égalise, parceque, dans l'exemple actuel, je puis raisonner sur la chose que me présente le double signe $+\sqrt{-1} \; -\sqrt{-1}$, comme sur celle que le signe o me présente, et que l'un et l'autre de ces signes me conduisent aux mêmes conséquences. Cette équation n'est pas réelle. Elle n'est qu'artificielle, comme tout l'est dans l'algèbre. Elle veut dire ceci: *un mois dont on fait abstraction est (relativement aux conséquences) égal à un mois qui n'existe pas.* Dans $\left(\frac{-i\sqrt{-1}}{2}, \frac{i\sqrt{-1}}{2}\right)$, c'est *la qualité de passé ou de futur* qui est zéro; dans o, c'est *la quantité de passé et de futur* qui l'est.

26. Le signe o a deux significations. On peut en effet le considérer sous un point de vue *arithmétique* et sous un point de vue *descriptif*. Sous le premier, o signifie *quantité nulle*. Sous le second, il signifie une description telle que la *distance* entre le premier et le dernier point est nulle. (Voyez la Figure 3).

27. Cet exemple fait voir, si je ne me trompe, que les signes $+\sqrt{-1}$ et $-\sqrt{-1}$ peuvent avoir une signification toutes les fois que les qualités représentées par les signes opposés + et − sont telles qu'une unité peut avoir l'*une* ou l'*autre* ou *ni* l'*une ni* l'autre.

De cette manière les signes + et − peuvent, même comme signes d'addition et de soustraction, admettre au milieu d'eux le signe $\sqrt{-1}$. Alors $+q$ signifie que la quantité q a la

qualité d'être *additive* ; $-q$, qu'elle est *soustractive* ; $q\sqrt{-1}$, qu'elle n'est *ni additive ni soustractive* ; que, par conséquent, $q\sqrt{-1}$ est étranger à l'équation qui contient $+q$ ou $-q$.

N°. Il faut bien remarquer ici qu'*être étranger* ne signifie pas *être nul*, mais *être regardé comme nul*. Dans l'exemple présent, *être étranger* signifie : *ni additif ni* soustractif. *Être nul* signifie : additif *et* soustractif *en même temps*.

28. 3°. Lorsque $\sqrt{-1}$ signifie une *qualité* de la chose prise pour unité, examinons ce que la présence de ce signe indique, dans le cas où la question demande qu'on fasse abstraction de cette qualité et qu'en conséquence on ne fasse pas usage de son signe. Quand on fait une abstraction, comme par exemple, quand, en considérant la *longueur* d'une ligne, on fait abstraction de sa *largeur*, il faut que tous les raisonnemens qu'on fait sur cette ligne soient indépendans de cette *largeur*. Il faut par conséquent que le signe de cette largeur ne paroisse pas, ou qu'il soit accompagné, non du signe $\pm$, mais des signes séparés $+$ et $-$ dont l'un puisse détruire l'effet de l'autre. Si $\sqrt{-1}$ exprime cette *largeur* et qu'il paroisse dans le résultat de ces raisonnemens, sans pouvoir être détruit autrement que par une nouvelle équation qui contredise les équations établies, il indique quelque contradiction dans ces raisonnemens. Il indique donc alors une absurdité. Lors donc qu'on trouve ce signe dans un résultat, il faut voir si la question exigeoit qu'on fît abstraction de la qualité indiquée par ce signe. Si elle l'exigeoit, la question étoit impossible. Sinon, elle ne l'étoit pas.

29. Si elle exigeoit qu'on fît cette abstraction, cette abstraction étoit une condition. L'expression de cette condition devoit être une équation. Cette équation nouvelle supposoit

une nouvelle indéterminée pour que le problème ne fût pas plus que déterminé.* Or (No. 14.) l'unité renfermée dans le signe $\sqrt{-1}$ pouvant être regardée comme une nouvelle indéterminée introduite par ce signe, et son indétermination étant relative non à la *quantité*, mais à la *qualité*, la seule manière d'exprimer qu'on faisoit abstraction de cette qualité, étoit d'égaler à zéro tous les termes multipliés par $\sqrt{-1}$. Si donc le résultat donne la somme de ces termes comme n'étant pas $=0$, il est en contradiction avec les conditions de la question.

30. 4°. Mais *toutes les fois qu'on peut donner au signe* $\sqrt{-1}$ *une signification compatible avec les conditions du problème*, c'est-à-dire, lorsque ces conditions n'admettent ni n'excluent cette signification, l'indétermination qui accompagne l'unité qu'il renferme corrige le défaut que ce problème peut sembler avoir d'être plus que déterminé.

31. De tout ce qui vient d'être dit je conclus que partout où l'algèbre n'est qu'*arithmétique universelle* et où le signe $\sqrt{-1}$ se trouve mêlé avec les signes $+$ ou $-$ sans pouvoir être supprimé, les quantités qui portent ce signe $\sqrt{-1}$ sont des *quantités imaginaires*; mais que, quand l'algèbre devient une *langue* (et elle le devient dans toute équation, puisque toute équation est une proposition) alors les quantités qui portent le signe $\sqrt{-1}$ peuvent être ou n'être pas réelles. Elles le sont lorsque ni la *qualité* figurée par le signe $\sqrt{-1}$, ni la *quantité* affectée de ce signe, ne sont en contradiction avec les conditions de la question.

* *Voyez la Géométrie de Position de* Mr. Carnot (page 53.)

32. Je crois qu'il est à propos d'ajouter ici une remarque analogue à celle du No. 8.

Dans ce No. je dis que les deux significations de + et — ne peuvent pas avoir lieu en même temps, relativement au même + ou au même —. J'en dis autant du signe $\sqrt{-1} \times \sqrt{-1}$. Mais cette remarque a besoin d'être développée.

Principe général. Lorsqu'on a une équation identique, les signes +, — et $\sqrt{-1}$ ne peuvent pas avoir une signification dans un de ses membres et un autre signification dans les termes semblables de l'autre membre.

Par exemple, si l'on fait (Fig. 1.) AB = 1 et AD = $\sqrt{-1}$, on aura $\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = 1^2 + (\sqrt{-1})^2$. Supposons pour un instant $1^2 + (\sqrt{-1})^2 = +1-1 = 0$. On aura $\overline{BD}^2 = 0$, ce qui est absurde. C'est que le premier membre $1^2 + (\sqrt{-1})^2$ ou $\overline{AB}^2 + \overline{AD}^2$ représente une *figure* formée de deux carrés égaux tels que les côtés de l'un sont *perpendiculaires* aux côtés correspondans de l'autre (No. 11.); tandis que le second membre +1 —1 ou 0 signifie la *différence* de deux unités abstraites égales. Ainsi l'équation $\overline{BD}^2 = 0$ peut être traduite par cette proposition : la *figure* $\overline{BD}^2$ est égale à la *différence de deux unités abstraites*. Cette proposition ne renferme point de contradiction, mais elle ne présente aucun sens. Les idées qu'elle allie ne sont point *opposées*, mais *disparates*.

33. Il est bon de faire attention à cette distinction entre *opposé* et *disparate*. Si les signes + et — n'avoient qu'une signification, le signe — auroit nécessairement la même signification dans les deux membres de l'équation $\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = \overline{AB}^2 - (-\overline{AD}^2) = 1^2 - 1^2 = 0$. Cette équation offriroit

deux idées *opposées* regardées comme une seule et même idée. Elle seroit contradictoire. *Ce seroit la faute du signe* −. Mais si les signes + et − ont chacun deux significations, l'équation $\overline{AB}^2 + \overline{AD}^2 = 1^2 - i^2 = 0$ n'offre plus que deux idées disparates. Elle est l'effet d'une confusion d'idées. *C'est la faute de l'analiste.*

34. Cette distinction me paroit prévenir les objections insurmontables que Mr. Carnot fait, dans le discours préliminaire de sa *Géometrie de Position*, contre cette proposition : les signes + et − indiquent deux directions opposées. Ses objections supposent tacitement que ces signes n'ont qu'une signification.

35. Autre exemple. Selon la manière ordinaire de s'exprimer, on a (Fig. 1.) $\overline{BD}^2 = 1^2 + 1^2$. Par conséquent $\overline{BD} = (1+\sqrt{-1^2})(1-\sqrt{-1^2})$, et $1+\sqrt{-1^2} : \sqrt{2} :: \sqrt{2} : 1-\sqrt{-1^2}$, proportion absurde, si l'on attribue à $\sqrt{2}$ sa signification arithmétique. Mais si l'on multiplie la seconde raison de cette proportion par $1+\sqrt{-1^2}$, on aura $1+\sqrt{-1^2} : \sqrt{2} :: \sqrt{2} \times 1+\sqrt{-1^2} : 2$, ou $1+\sqrt{-1^2} : \sqrt{2} \times \sqrt{1+\sqrt{-1^2}} :: \sqrt{2} : \sqrt{2} \times \sqrt{2}$, proportion dont la vérité saute aux yeux.

J'ai dit : *si on attache à* $\sqrt{2}$ *sa signification arithmétique* ; car si l'on attache à $\sqrt{2}$ sa signification géométrique, qui est de représenter la diagonale d'un carré dont le côté est 1, alors la proportion $1+\sqrt{-1^2} : \sqrt{2} :: \sqrt{2} : 1-\sqrt{-1^2}$ ne sera plus absurde. En effet, si $\sqrt{2}$ représente $\overline{BD}$ (Fig. 1.), il représente une ligne dont la direction, par rapport à $\overline{BA}$, peut être représentée de la manière suivante : soit $\sqrt{-1} = 1 \times e^{\frac{\pi\sqrt{-1}}{2}}$

(e étant la base des logarithmes hyperboliques et π la demi-circonférence d'un cercle dont le rayon est 1); $1 \times e^{\frac{\pi}{2}\sqrt{-1}}$ signifie la ligne $\overline{AD}$ dont la direction est $e^{\frac{\pi}{2}\sqrt{-1}}$. De même $\sqrt{2} \times e^{\frac{\pi}{4}\sqrt{-1}}$ signifiera la ligne $\overline{BD}$ dont la direction est $e^{\frac{\pi}{4}\sqrt{-1}}$, c'est-à-dire, demi-perpendiculaire. Alors la proportion précédente deviendra $1+1 \cdot e^{\frac{\pi}{2}\sqrt{-1}} : \sqrt{2} \cdot e^{\frac{\pi}{4}\sqrt{-1}} :: \sqrt{2} \cdot e^{-\frac{\pi}{4}\sqrt{-1}} : 1-1 \cdot e^{\frac{\pi}{2}\sqrt{-1}}$, ou bien $1+\sqrt{-1} : \sqrt{2}(\cos. 45^\circ + \sin. 45^\circ \sqrt{-1}) :: \sqrt{2}(\cos. 45^\circ - \sin. 45^\circ \sqrt{-1}) : 1-\sqrt{-1}$; ou bien $1+\sqrt{-1} : \sqrt{2}\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right) :: \sqrt{2}\left(\frac{1-\sqrt{-1}}{\sqrt{2}}\right) : 1-\sqrt{-1}$, qui est identique.

36. Pour répandre sur cette matière autant de clarté qu'il m'est possible, je me proposerai quelques questions à résoudre. La première sera prise de l'ouvrage déjà cité de Mr. Carnot. J'exposerai d'abord la solution qu'il en donne, ensuite la mienne. Ce rapprochement rendra plus sensibles les principes que je m'efforce d'établir dans ce Mémoire.

Problème II.

37. Voici les termes de Mr. Carnot (No. 58.)

" Proposons-nous cette question : une droite $\overline{AB}$ (Fig. 4.) " étant donnée, trouver sur cette droite un point K, tel que " le produit des deux segmens $\overline{AK}$, $\overline{BK}$, soit égal à une quan- " tité donnée; par exemple, à la moitié du carré de $\overline{AB}$.

" Comme je ne sais encore si le point K doit se trouver sur " la droite même $\overline{AB}$, ou sur son prolongement, j'établis

" d'abord mon calcul, en supposant que c'est sur la droite " même ; c'est-à-dire, que K tombe entre A et B.

" Cela posé, prenant $\overline{AK}$ pour l'inconnue, je la désigne par " x, et je nomme a, la droite donnée $\overline{AB}$; la condition du " problème me donnera donc $x(a-x)=\frac{1}{2}a^2$, ou x^2-ax+ " $+\frac{1}{2}a^2=0$, d'où je tire $x=\frac{1}{2}a\pm\sqrt{-\frac{1}{4}a^2}\left[=\frac{a}{2}(1\pm\sqrt{-1})\right]$, " c'est-à-dire, que x est imaginaire.

" Je ne conclus pas de là que la solution du problème pro- " posé soit impossible ; mais seulement qu'elle l'est dans la " supposition que j'ai faite, que le point K est placé entre A et " B ; c'est-à-dire, que le problème a pu être mal mis en équa- " tion, parce que j'aurai établi mes raisonnemens sur une " figure qui n'étoit pas celle que je devois considérer, ou qui " ne pouvoit satisfaire aux conditions du problème. J'établis " donc de nouveau mon raisonnement, en partant d'une hy- " pothèse autre que celle que j'avois faite d'abord, c'est-à- " dire, que je supposerai le point cherché, non sur $\overline{AB}$, comme " je l'avois fait, mais sur un de ses prolongemens, par ex- " emple, en K'.

" Alors la condition du problème me donne $x(x-a)=\frac{1}{2}a^2$, " ou $x^2-ax-\frac{1}{2}a^2=0$; d'où je tire $x=\frac{1}{2}a+\sqrt{\frac{3}{4}a^2}$, équation " qui ne contenant plus d'imaginaires résout la question pro- " posée.

" Cette solution est double ; l'une $x=\frac{1}{2}a+\sqrt{\frac{3}{4}a^2}$ étant " positive, résoud sans difficulté la question, conformément à " ma nouvelle hypothèse, c'est-à-dire, en supposant que le " point cherché est sur le prolongement de $\overline{AB}$, au de là du " point B ; ou que le point B se trouve entre A et le point " cherché. Mais l'autre solution $x=\frac{1}{2}a-\sqrt{\frac{3}{4}a^2}$ étant né-

" gative, ne peut se rapporter à la même hypothèse, et " d'après ce qui a été dit ci-dessus, il faut, pour en avoir la " signification, changer le signe, et voir à quel système corré- " latif l'équation ainsi modifiée pourra satisfaire. Or de ce " changement il résulte, que l'équation $x(x-a)=\frac{1}{4}a^2$, qui " exprime la condition du problème, devient $x(x+a)=\frac{1}{4}a^2$. " Voyons à quel nouveau système corrélatif peut satisfaire " cette nouvelle expression de la condition du problème.

" Or il est facile de voir que c'est en supposant que le point " K tombe sur le prolongement de $\overline{AB}$, non du côté de B, " comme ci-dessus, mais du côté de A en K". Et en effet, " en partant de cette nouvelle hypothèse, x sera $\overline{AK''}$, et $\overline{BK''}$ " sera $x+a$; d'où il suit que la condition du problème sera " $x(x+a)=\frac{1}{4}a^2$, et cette équation donnera $x=-\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2}$ " dont la racine positive est effectivement la même que celle " qui s'étoit présentée négativement."

98. Dans cette solution, Mr. CARNOT raisonne rigoureusement d'après les principes fondamentaux de l'algèbre considérée comme *arithmétique universelle*. Ces principes n'admettant que des unités abstraites ne peuvent admettre de quantités négatives par elles-même, et à plus forte raison, de *carrés négatifs*. Conséquemment à ces principes, Mr. CARNOT est obligé de changer sa première équation en une seconde qui en diffère essentiellement, et celle-ci en une troisième qui ne diffère de la seconde qu'en apparence. En effet sa première équation est $-x^2+ax-\frac{1}{2}a^2=0$ (1);
sa seconde - - - $+x^2-ax-\frac{1}{2}a^2=0$ (2),
et sa troisième - - $+x^2+ax-\frac{1}{2}a^2=0$ (3).

39. Or je dis que la première diffère *essentiellement* de la seconde. Si l'on met dans l'une et dans l'autre y^2 à la place de o, on aura $-x^2+ax-\frac{1}{2}a^2=y^2$ - - - (4), et - $+x^2-ax-\frac{1}{2}a^2=y^2$ - - - (5).

La première de ces équations appartient à un cercle dont le rayon est imaginaire et dont je donnerai bientôt la description. La seconde appartient à une hyperbole équilatère dont l'axe est $\frac{a}{2}\sqrt{3}$.

40. Je dis que la seconde ne diffère de la troisième qu'en apparence. En effet ces deux équations donnent (en y mettant y^2 au lieu de o) - - - $x^2-ax-\frac{1}{2}a^2=y^2$ - - (5);

$x^2+ax-\frac{1}{2}a^2=y^2$ - - (6).

Elles appartiennent à la même hyperbole équalitère dont les axes sont $=\frac{a}{2}\sqrt{3}$. La première exprime les deux branches positives et sousentend les négatives. La seconde exprime les deux branches négatives et sousentend les positives.

41. Dans les principes de Mr. Carnot qui sont, je le répéte, les principes fondamentaux de l'algèbre considérée comme *arithmétique universelle*, l'équation (1) ne donne aucune solution; la seconde n'en donne qu'une, et la 3e. une.

Dans les principes que j'ai exposés et qui appartiennent à l'*algèbre-langue*, ces trois équations donnent chacune deux solutions. Les deux solutions données par la seconde sont les mêmes que celles données par la troisième. Ainsi l'on peut supprimer l'une des équations. L'équation restante donnera les deux solutions de Mr. Carnot. Quant à la première, il faut développer les deux solutions qu'elle donne dans mes principes.

42. Soit (Fig. 5.) AK$=+\frac{a}{2}$, BK$=-\frac{a}{2}$, KC (décrit sur le

plan de ce papier) $= +\frac{a}{2}\sqrt{-1}$, KD $= -\frac{a}{2}\sqrt{-1}$, KE (décrit perpendiculairement au plan de ce papier) $= +\frac{a}{2}\sqrt{-1}$, KG $= -\frac{a}{2}\sqrt{-1}$.

CEDG est un cercle décrit du centre K et du rayon $\frac{a}{2}\sqrt{-1}$, perpendiculairement au plan de ce papier.

Le même cercle peut être décrit d'un mouvement conique par l'apothème AE dont une extrémité seroit fixe au point A. Ce cercle CEDG ainsi décrit est celui qui satisfait à l'équation (4).

48. En effet changeons, dans cette équation, x en $\left(x+\frac{a}{2}\right)$, c'est-à-dire, transportons l'origine des x, de A en K. Cette équation deviendra $-x^2-\frac{a^2}{4}=y^2$, ou $y^2=\left(\frac{a}{2}\sqrt{-1}\right)^2-x^2$ (7).

Elle représente le cercle CEDG décrit du centre K et du rayon $\frac{a}{2}\sqrt{-1}$.

Remettons l'origine des abscisses en K. Nous reviendrons à l'équation (4), et les abscisses seront $x=\frac{a}{2}+z$. Or $\frac{a}{2}=$AK est décrit sur le plan de ce papier. z au contraire étant une partie du rayon KE $=\frac{a}{2}\sqrt{-1}$, ou KG $=-\frac{a}{2}\sqrt{-1}$, est perpendiculaire à AK. Donc l'abscisse x est une ligne brisée formée de deux lignes perpendiculaires l'une à l'autre. Donc le nouveau rayon est la ligne brisée AKE. Si l'on fait tourner ce rayon sur AK, on aura le cercle CEDG. Or faire tourner le rayon AKE sur AK, c'est faire tourner en même temps AE d'un mouvement conique. Donc, comme je l'ai dit, le cercle CEDG décrit d'un mouvement conique par l'apothème AE, dont une extrémité est fixe au point A, est celui qui satisfait à l'équation (4).

44. Dans cette équation (4), y est une ordonnée prise sur le rayon KC $= +\frac{a}{2}\sqrt{-1}$, ou KD $= -\frac{a}{2}\sqrt{-1}$. Si l'on suppose $y = 0$, alors on a

$$x = \text{AKE} = \frac{a}{2} + \frac{a}{2}\sqrt{-1} = \frac{a}{2}(1+\sqrt{-1}),$$

ou $x = \text{AKD} = \frac{a}{2} - \frac{a}{2}\sqrt{-1} = \frac{a}{2}(1-\sqrt{-1})$.

Ces deux valeurs de x sont les deux racines de l'équation (1) qui est la première de Mr. Carnot.

45. Pour voir maintenant comment elles résolvent son problème, il suffit de l'énoncer de la manière suivante :

Une droite $\overline{\text{AB}}$ (Fig. 5.) *étant donnée, trouver sur cette droite un point* K (projection de la ligne $\overline{\text{KE}}$ sur la ligne $\overline{\text{AB}}$) *tel que le produit des* (deux lignes $\overline{\text{AE}}$, $\overline{\text{BE}}$ dont les) *deux segments* $\overline{\text{AK}}$, $\overline{\text{BK}}$ (sont les projections) *soit égal à la moitié du carré de* $\overline{\text{AB}}$.

Il suffit donc, pour faire cadrer l'équation (1) avec la question proposée, d'en regarder les *données* comme des *projections*, et d'en rapporter les *demandes*, non à ces projections, mais aux lignes originales.

On pourroit aussi, sans rien ajouter à l'énoncé de Mr. Carnot, supposer que le point K n'est un point que par rapport au plan de ce papier, c'est-à-dire que, quoiqu'il n'ait ni *longueur* ni *largeur sur* le plan de ce papier, il a une *hauteur au dessus*, et qu'ainsi, à quelque partie de cette hauteur que les lignes $\overline{\text{AE}}$ et $\overline{\text{BE}}$ puissent se joindre, elles sont censées se joindre à ce qu'on nomme le point K. Alors, dans la description de ce point, on ne feroit abstraction que des deux dimensions qui se trouvent sur le plan de ce papier, sans faire abstraction de celle qui en es dehors (No. 28). Dans ce cas

on pourroit retrancher de l'énoncé précédent toutes les parenthèses que j'y ai mises et qui sont des additions faites à l'énoncé de Mr. Carnot.

46. Pour montrer maintenant que les lignes $\overline{AE}$, $\overline{BE}$ résolvent la question, c'est-à-dire, que leur produit est égal à la moitié du carré de $\overline{AB}$, il y a une observation à faire.

Ce n'est pas le produit *des lignes* $\overline{AE}$, $\overline{BE}$, mais le produit de leurs *valeurs arithmétiques* qui résoud la question; car *un produit de lignes*, c'est-à-dire, un résultat de *lignes multipliées par des lignes* ne signifie rien. On ne demande pas une *figure géométrique*, mais un *nombre*. Or, pour avoir les valeurs arithmétiques de $\overline{AE}$, $\overline{BE}$, il faut écarter de leurs expressions les signes qui n'ont trait qu'à leurs positions. Sans cette précaution, on confondroit les *signes des valeurs numériques* avec des *signes de position* ou des signes purement descriptifs. Cela posé, il est clair que $\overline{AE}=\overline{BE}=\sqrt{\overline{AK}+\overline{KE}}=\sqrt{2\overline{AK}}=\overline{AK}\times\sqrt{2}$. Donc $\overline{AE}\times\overline{BE}=2\overline{AK}=2\left(\frac{\overline{AB}}{2}\right)^2=\frac{\overline{AB}^2}{2}$.

Problème III.

47. Quel est le point où se joindront les extrémités D, E, des lignes $\overline{AD}$, $\overline{BE}$ (Fig. 6.) tirées des extrémités A, B, de la ligne $\overline{AB}$, en supposant que la longueur de $\overline{AB}$ soit $2a$, celle de $\overline{AD}$, $\frac{1}{2}a$, et celle de $\overline{BE}$, aussi $\frac{1}{2}a$?

Cette question paroît évidemment absurde. Résolvons-la.

Soit $\overline{Da}=x$, $\overline{Aa}=y$, $\overline{Eb}=x'$, et $\overline{Bb}=y'$. On a $\overline{AD}^2=\frac{1}{4}a^2=x^2+y^2$, et $\overline{BE}=\frac{1}{4}a^2=x'^2+y'^2$.

Or dire que les points D et E doivent *se joindre*, ou dire qu'ils doivent *toucher le même point*, c'est dire la même chose.

Il en est de même des points a et b qui doivent être l'un et l'autre sur le point C, milieu de la ligne $\overline{AB}$. On a donc $\overline{Aa} = y = a$, et $\overline{Bb} = y' = a$. Par conséquent $\overline{AD}^2 = \frac{1}{4}aa = x^2 + a^2$, et $\overline{BE}^2 = \frac{1}{4}aa = x'^2 + aa$. Ces deux équations donnent $x = x' = \pm \frac{\sqrt{-3}}{2} \cdot a$, ou $x = \pm \frac{\sqrt{3}}{2} \cdot a\sqrt{-1}$, quantité imaginaire, comme on devoit bien s'y attendre.

48. Montrons cependant que cette quantité imaginaire, indique un sens raisonnable dont la question proposée est susceptible.

Il est certain que, tant qu'on regardera les lignes $\overline{AD}$, $\overline{BE}$, comme des lignes étendues seulement en longueur, sans aucune largeur, et le point avec lequel elles doivent coïncider, comme un point sans extension, la question sera impossible (No. 28).

Mais le signe $\sqrt{-1}$ que renferme la solution indique ce qui peut la rendre possible, en indiquant une *largeur* dans les lignes $\overline{AD}$, $\overline{BE}$, ou une *extension* dans le point C.

49. En effet, ce signe $\sqrt{-1}$ qui est attaché à x dont la valeur est $\sqrt{3} \times \frac{a}{2}$, ou $\sqrt{3} \times \overline{AD}$, montre que x doit être perpendiculaire à $\overline{AD}$ (No. 10).

D'après cela, on peut supposer que les lignes $\overline{AD}$ et $\overline{BE}$, dont la longueur est $\frac{a}{2}$, ont une largeur $= \sqrt{3} \times \frac{a}{2}$. Dans cette supposition, ces lignes deviennent des rectangles $\overline{\overline{ADCF}}$, $\overline{\overline{BDCF}}$,* qui se joignent au point C. Dans cette même

* J'adopte ici, et dans le courant de ce Mémoire, la notation de Mr. Carnot. Dans cette notation les lignes sont désignées par des lignes mises au dessus des lettres qui indiquent leurs extrémités. Les parallélogrammes le sont par de doubles lignes, les angles par des lignes brisées, et les courbes par des lignes courbes.

supposition, quoique les extrémités D et E des lignes $\overline{AD}$ et $\overline{BE}$ ne se joignent pas, elles sont les projections du point de jonction C sur les lignes $\overline{AD}$ et $\overline{BE}$ dont elles sont les extrémités.

50. On peut encore supposer que le point de jonction, au lieu d'être sans extension, en a une, et qu'au lieu d'être un cercle infiniment petit dont le rayon est 0, c'est un cercle fini dont le rayon est $\sqrt{3}\frac{a}{2}$. Dans cette supposition, les lignes sans largeur $\overline{AD}$, $\overline{BE}$ (Fig. 7.) sont tangentes au même cercle DHEE'KD (regardé comme faisant la fonction d'un point) puisqu'elles sont perpendiculaires aux extrémités de ses rayons $\overline{CD}$, $\overline{CE}$.

51. On peut réunir les deux suppositions précédentes. Par exemple, on peut donner aux lignes $\overline{AD}$, $\overline{BE}$, les largeurs $\overline{Dd}$, $\overline{Ee}$, prises à volonté, et attribuer au point C l'étendue du cercle $\overset{\frown}{dhee'kd'}$. Dans cette nouvelle supposition, les rectangles $\overline{ADdf}$, $\overline{BEeg}$, sont tangens au cercle $\overset{\frown}{dhee'kd'}$.

52. Enfin, au lieu de rectangles, on pourroit supposer des cylindres, et au lieu de cercles, des sphères.

53. Les descriptions que je viens de donner n'ont rapport qu'à la valeur positive de x ou de x', qui est $+\frac{\sqrt{-3}}{2}.a$. Pour avoir celles qui ont rapport à sa valeur négative $-\frac{\sqrt{-3}}{2}.a$, il suffit de transporter au dessous de la ligne $\overline{AB}$ tout ce qui se trouve au dessus, dans la Figure 7, *et vice versâ*.

Problème IV.

54. Soient deux plans carrés. Que le côté de l'un excède le côté de l'autre, de deux piés, et que le nombre des piés carrés contenus dans les deux pris ensemble soit 1. Quelles sont les dimensions de ces deux plans carrés?

Je désigne par x le côté d'un de ces carrés, et par $x+2$ le côté de l'autre. On a donc, par la question, l'équation $x^2+(x+2)^2=1$, ou $x^2+2x+2=\frac{1}{2}$, qui donne $x=-1\pm\sqrt{-\frac{1}{2}}$.

Ces valeurs de x sont imaginaires. La question qu'elles résolvent est impossible, si on suppose en même temps ces carrés sans épaisseur et sans vides. Mais si on leur suppose quelque épaisseur ou quelque vide, alors la question n'est plus impossible.

55. Voyons ce que nous indiquent les signes $+$, $-$, $\pm\sqrt{-1}$. On peut les interpréter, soit géométriquement, soit arithmétiquement.

56. Signes interprétés géométriquement.

Le côté de l'un de ces carrés est $-1\pm\sqrt{-\frac{1}{2}}$, et le côté de l'autre, $+1\pm\sqrt{-\frac{1}{2}}$. -1 et $\pm\sqrt{-1}$ indiquent deux lignes dont l'une est perpendiculaire à l'autre. Par conséquent si $(-1\pm\sqrt{-\frac{1}{2}})$ indique une seule ligne, l'une des deux quantités qui composent $(-1\pm\sqrt{-\frac{1}{2}})$ indique la *longueur* de cette ligne, et l'autre, l'*épaisseur de son point extrême*. Elles ne peuvent pas exprimer, l'une la *longueur* et l'autre la *largeur*, parcequ'il est de l'essence du carré exprimé algébriquement que sa *longueur* et sa *largeur* aient la même expression. Il en est de même de $+1\pm\sqrt{-\frac{1}{2}}$.

Supposons que $\pm\sqrt{-\frac{1}{2}}$ soit la *longueur* de ces côtés. Alors $+1$ sera l'*épaisseur du point extrême*, dans un des carrés demandés, et -1 l'*épaisseur de ce point*, dans l'autre. Soit (Fig. 8.) $\overline{AB} = \overline{A'B} = 1$. On aura $\overline{AA'} = \overline{AB} = \overline{A'B} = \overline{BB'} =$ (arithmétiquement) $\sqrt{\frac{1}{2}}$. La *longueur* du côté cherché sera donc $\overline{AA'}$, et l'*épaisseur de son point extrême*, $\overline{AB'}$.

57. Il s'agit de savoir si ces valeurs résolvent la question proposée.

La première condition de cette question est que le côté d'un des carrés excède le côté de l'autre, de deux pieds. Pour savoir de combien une ligne surpasse une autre ligne, il faut les poser l'une sur l'autre. Ce qui reste découvert est l'excédent. Lorsque ces lignes sont les côtés de deux carrés, ce sont les carrés qu'il faut poser l'un sur l'autre.

Faisons donc cette opération. Pour cela j'élève le plan du carré $ABA'B'$ dont le côté est $+\sqrt{-\frac{1}{2}}$ ou $-\sqrt{-\frac{1}{2}}$ perpendiculairement au plan de ce papier, de sorte que son côté $\overline{AB}$ se trouve sur ce papier même, comme on le voit en $\overline{AB}$, Figure 9. J'applique ensuite, par la pensée, le plan du second carré sur celui du premier. Ce second plan est aussi $= ABA'B'$, et son côté $= \overline{AB}$. La Figure 9. nous montre les côtés $\overline{AB}$ de nos carrés appliqués l'un sur l'autre. *L'épaisseur du point extrême* de l'un est $\overline{Aa} = \overline{Bb} = +1$, et *celle du point extrême* de l'autre, $\overline{Aa'} = \overline{Bb'} = -1$. Ces épaisseurs restent à découvert. Leur somme est $+1+1=2$; car les signes $+$ et $-$ que portent les valeurs de ces épaisseurs représentent leurs *directions*, puisqu'ils sont interprétés géométriquement. Ils ne signifient pas que l'une doive être ôtée de l'autre.

58. Ce n'est donc pas *la différence des longueurs* des côtés qui est égale à 2, comme le demande la question, mais *la somme des épaisseurs de leurs points extrêmes*. Cette solution peut cependant se rapprocher de ce que demande la question, en la considérant ainsi: l'excédent du côté d'un carré sur l'autre est égal à la différence de position de l'une de leurs extrémités, tandis que l'autre extrémité coïncide, dans les deux carrés. Ainsi, dans la Figure 9, les extrémités B et B coïncident, et les extrémités a et a' sont distantes l'une de l'autre, d'une longueur $= \overline{aa'}$. La question demande que cette distance soit $= 2$, relativement à la *longueur* des côtés. La solution la donne $= 2$, relativement à l'*épaisseur de leurs points extrêmes*, c'est-à-dire, relativement à une seconde *longueur* perpendiculaire à la première.

59. La seconde condition de nôtre question est que le nombre des piés carrés contenus dans les deux carrés pris ensemble soit 1. Or l'*épaisseur* des côtés ne doit entrer pour rien dans ce nombre de piés carrés, puisque des piés carrés sont des surfaces telles que, pour les mesurer, on ne prend la mesure que dans deux dimensions.

Cela posé, le côté d'un des carrés est $+1 \pm \sqrt{-\frac{1}{2}}$ (en y comprenant la longueur et l'épaisseur, comme on le doit, puisqu'ils ont une épaisseur) et le côté de l'autre est $-1 \pm \sqrt{-\frac{1}{2}}$. Les deux carrés sont donc

$$(-1 \pm \sqrt{-\tfrac{1}{2}})^2 = \tfrac{1}{2} \mp 2\sqrt{-\tfrac{1}{2}} = \tfrac{1}{2} \mp \sqrt{-2},$$

$$\text{et } (+1 \pm \sqrt{-\tfrac{1}{2}})^2 = \tfrac{1}{2} \pm 2\sqrt{-\tfrac{1}{2}} = \tfrac{1}{2} \pm \sqrt{-2}.$$

Dans ces carrés, $\frac{1}{2}$ et $\frac{1}{2}$ sont *les carrés des longueurs*; $\mp 2\sqrt{-\frac{1}{2}}$ et $\pm 2\sqrt{-\frac{1}{2}}$ sont *les sommes des épaisseurs des côtés de ces carrés*.

Par conséquent $\frac{1}{2}$ et $\frac{1}{2}$ sont les faces. Donc *la somme des faces* des deux carrés est 1, comme la question le demande.

60. *Remarques.* 1°. On a vu que, dans la mesure des côtés, ce sont *les épaisseurs des points extrêmes*, et dans celle des carrés, les *faces* qu'il faut considérer. Cette circonstance est indiquée par les expressions de ces *côtés* et de ces *carrés*. Les expressions des côtés (qui sont $\frac{1}{2} \pm \sqrt{-\frac{1}{2}}$ et $\frac{1}{2} \pm \sqrt{-\frac{1}{2}}$) indiquent que les *longueurs* ($\pm \sqrt{-\frac{1}{2}}$) sont des perpendiculaires étrangères à la question, et que *les épaisseurs des points extrêmes* n'en sont pas. Les expressions des *carrés* (qui sont $\frac{1}{2} \pm 2\sqrt{-\frac{1}{2}}$ et $\frac{1}{2} \mp 2\sqrt{-\frac{1}{2}}$) indiquent que les *épaisseurs des côtés* sont imaginaires, et deviennent par là les perpendiculaires étrangères à la question, et que les *faces* sont réelles, comme elles doivent l'être, puisqu'elles sont l'objet de la question. C'est ce que représentent les Figures 9 et 10. Dans la première, les *épaisseurs des points extrêmes* ($\overline{Aa}$, $\overline{A'a'}$) sont sur le plan de ce papier. Dans la seconde, ce sont les *faces* ($\overline{ABA'B'}$). Or, dans une question où il s'agit de mesurer des lignes et des surfaces qui sont censées décrites sur un même plan, on ne doit mesurer que celles qui sont sur ce plan supposé.

61. 2°. Les côtés des carrés sont les lignes brisées $\widehat{aAB}$, $\widehat{a'AB}$ (Fig. 9). Les carrés même sont les plans brisés $\widehat{a'A'AabBB'}$, $\widehat{a''A'Ac'b'BB'}$ (Fig. 10). Les lignes brisées présentent deux dimensions. Les plans brisés en présentent trois.

62. 3°. Dans la Figure 10, les plans qui expriment les épaisseurs des côtés ne sont établis que sur deux des côtés du

carré. Mais on pourroit les établir sur les quatre côtés sans que la solution du problème en souffrît. C'est une suite de l'indétermination dont il a été parlé au No. 15. Dans ce cas, les deux carrés que demande le problème se trouveroient former une boite dont ils seroient les deux moitiés. Cette boite ne contiendroit que des plans sans épaisseur. Les épaisseurs des côtés des carrés demandés seroient les hauteurs des ces deux moitiés de boite. Les carrés demandés seroient ses deux fonds, dont l'un supérieur et l'autre inférieur.

63. Signes $+$, $-$ et $\pm\sqrt{-1}$ interprétés arithmétiquement.

$\pm\sqrt{-1}$ n'a aucune signification arithmétique (No. 10); mais $\pm\sqrt{-1}\times\pm\sqrt{-1}$ $(=-1^{\circ})$ en a une. -1° est un *carré soustractif*, et par conséquent formant *un vide* au milieu d'une surface pleine. De là il suit que, lorsqu'il s'agit de carrés et, en général, de plans, le signe $\pm\sqrt{-1}$ peut indiquer des vides, au lieu de perpendiculaires.

64. Mais les perpendiculaires ne diminuent ni n'augmentent les longueurs des lignes auxquelles elles sont perpendiculaires. Si donc on substitue des *vides* aux perpendiculaires et des *pleins* aux lignes auxquelles elles sont perpendiculaires, ces vides ne doivent pas diminuer ces pleins. Il faut donc faire une *addition de vide*, et non une *soustraction* de plein. Or ajouter des vides à une surface, c'est augmenter ses dimensions sans augmenter le nombre des piés carrés qu'elle contient; c'est l'augmenter *géométriquement*, sans l'augmenter *arithmétiquement*. De même, ajouter $+a\sqrt{-1}$ $-a\sqrt{-1}$ à une quantité qui se *mesure* et qui se *compte*, c'est ajouter à sa *mesure* sans ajouter à sa *somme*. En effet, les signes $+$ et $-$

pris *géométriquement* indiquent des directions opposées, *en partant du même point*, et par conséquent des directions qui s'ajoutent; au lieu que ces mêmes signes pris *arithmétiquement* se détruisent l'un l'autre.

65. La question proposée contient deux parties, l'une géométrique, l'autre arithmétique. La partie géométrique est *la disposition* des lignes et des surfaces (No. 2). La partie arithmétique est *le calcul* de nombre des piés linéaires contenus dans les côtés qui sont des lignes, et de celui des piés carrés contenus dans les carrés même qui sont des surfaces.

Or, dans cette seconde solution, où les signes $+$, $-$, $\pm\sqrt{-1}$ sont pris arithmétiquement, il ne peut être question que de la partie arithmétique du problème. Je puis donc ajouter tant de vides que je voudrai, puisqu'ils n'altèrent pas la solution arithmétique. Je puis donc résoudre la question proposée en réduisant les deux surfaces carrées à des cadres carrés tels, que le côté de l'un surpasse le côté de l'autre, de deux piés, et que chacun de ces cadres ait une surface égale à un demi pié-carré.

66. Il y a une infinité de manières de résoudre la question proposée, au moyen de cadres de cette espèce, puisqu'on n'a que deux conditions à remplir et quatre variables pour les remplir. Les deux conditions sont, la différence des côtés qui doit être $=2$, et la somme des carrés qui doit être $=1$. Les quatre variables sont, 1°. le côté du premier carré, 2°. celui du second, 3°. le côté du carré vide renfermé par l'un des cadres, 4°. le rapport entre la surface du premier cadre et celle du second. Dans la solution, on a fait ces surfaces égales chacune à la moitié d'un pié carré; mais rien n'oblige à les faire égales.

67. *Remarques.* 1°. La différence entre les deux solutions précédentes consiste en ce que, dans la seconde, la différence des côtés est exprimée par *la différence de leurs longueurs* qui est une quantité arithmétique; au lieu que, dans la première, elle est exprimée par *la différence de situation* de ces côtés qui est une quantité géométrique (No. 2).

68. 2°. On pourroit réunir les deux solutions précédentes et résoudre le problème au moyen d'une boite carrée dont les deux fonds contiendroient en somme un pié carré de *plein* et tant de *vides* qu'on voudroit, et dont les hauteurs seroient chacune d'un pié avec *tant et si peu de plein* qu'on jugeroit à propos.

On voit par là, pour le dire en passant, combien la question proposée qui sembloit précise l'étoit peu. Il est certain qu'elle peut être entendue de toutes les manières qu'offrent les solutions précedentes. Si cependant on ne veut pas y reconnoître toutes ces significations, on doit voir du moins que les expressions de la langue algébrique qui correspondent à celles du langage ordinaire sont infiniment plus étendues que celles-ci. Elles ne sont pas vagues, puisqu'on peut en trouver toutes les significations, à l'aide de quelques principes; mais elles sont générales.

Problème V.

69. Un marbrier a deux cubes de marbre. Le côté d'un de ces cubes excède le côté de l'autre, de deux piés, et le nombre des piés cubes contenus dans les deux est 28. Quelles sont les dimensions de ces deux cubes?

Avant de donner la solution de cette question, j'ai une remarque à faire.

Cette question conduit à une équation du 3e. dégré. Toute équation du 3e. dégré a au moins une racine réelle. Par conséquent si, au lieu de 28, qui est le nombre des piés cubiques contenus dans les deux cubes, on n'avoit, par exemple, que 3 piés et $\frac{1}{4}$, on devroit encore avoir une solution possible. Cette solution donneroit pour le nombre des piés cubiques contenus dans un des cubes, $\frac{27}{8}$, et dans l'autre, $-\frac{1}{8}$. Or, pour que ce résultat qu'on appelle possible eût un sens raisonnable, il faudroit supposer qu'un des deux cubes fût un vide fait dans l'autre, c'est-à-dire, qu'il faudroit supposer un cube de $\frac{27}{8}$ pouces cubiques contenant un vide de $\frac{1}{8}$ de pouce cubique. Mais cette solution est toute semblable à celle qu'ont fournie les racines imaginaires de l'équation du problême précédent. Les deux solutions ont donc la même espèce de possibilité, quoique l'une soit donnée par un résultat imaginaire et l'autre par un résultat qui ne l'est pas.

La solution même du problême précédent répond mieux à l'énoncé de la question que la solution de celui-ci; car, dans la première de ces questions, on demande deux plans carrés, c'est-à-dire, deux étendues qui aient chacune deux dimensions égales. Or une étendue peut être vide. Dans la seconde question, au contraire, on demande deux cubes *de marbre*. Or un cube de marbre n'est pas un cube de vide.

7o. Revenons à nôtre 5e. problême.

Ce problême ne présente à l'esprit qu'une solution possible, et l'équation du 3e. dégré qui en exprime les conditions n'a qu'une racine réelle.

Cette équation est $x^3+(x+2)^3=28$ - - (7).

Ses trois racines sont - - - - - - -

$x=1$ . . . . . dont le cube est $x^3=1$;

$x=-2+\sqrt{-6}$ . . dont le cube est $x^3=28+6\sqrt{-6}$;

$x=-2-\sqrt{-6}$ . . dont le cube est $x^3=28-6\sqrt{-6}$.

Ces racines sont les côtés des cubes x^3.

Les côtés des cubes $(x+2)^3$ sont

$x+2=3$ . . dont le cube est $(x+2)^3=27$;

$x+2=+\sqrt{-6}$ dont le cube est $(x+2)^3=-6\sqrt{6}$;

$x+2=-\sqrt{-6}$ dont le cube est $(x+2)^3=+6\sqrt{-6}$.

En raisonnant sur les deux dernières des racines x et $\sqrt{x+2}$ comme on a raisonné sur les deux racines $-1\pm\sqrt{-\frac{1}{2}}$ qui résolvent la question précédente, on parviendra à des résultats semblables, l'un géométrique et l'autre arithmétique. Il y a cependant ici quelques remarques à faire.

71. 1°. Pour que le résultat géométrique soit juste, il faut que les mesures qu'il fournit remplissent les conditions de la question. Or si l'on raisonne relativement aux cubes comme on l'a fait (No. 60.) relativement aux carrés, on verra qu'il ne faut mesurer que ce qui ne porte pas le signe $\sqrt{-1}$. De plus, les mesures données par les termes réels sont justement ce qu'il faut pour satisfaire aux conditions de la question. Il ne faut donc pas mesurer les autres.

Mais il se présente ici une difficulté qui n'a pas lieu dans le cas du No. 60. C'est qu'un des cubes, savoir, $+\sqrt{-6}$ ou $-\sqrt{-6}$ est tout entier sous le signe $\sqrt{-1}$. Il ne faut donc en mesurer aucune partie. Il est donc étranger à la question. Il n'en remplit donc pas les conditions. Les conditions demandent deux cubes, et l'on n'en trouve qu'un.

Il faut convenir que, dans ce cas, les deux racines imaginaires

de l'équation ne donnent que des solutions impossibles, si on les considère géométriquement, c'est-à-dire, si l'on s'en tient aux lignes même et aux positions de ces lignes fournies par ces deux racines.

72. Elles peuvent cependant résoudre la question, même géométriquement, elles sont même les seules qui le puissent, si l'on énonce cette question de la manière suivante :

Un marbrier se propose de tailler deux cubes de marbre. Il veut que le côté d'un de ces cubes excède le côté de l'autre, de deux piés ; mais il ne peut y employer que 28 piés cubiques de marbre. Cette quantité n'étant pas suffisante pour la grandeur des cubes qu'il veut avoir, il est obligé d'y joindre de faux marbre pour remplir les vides que l'accroissement des dimensions doit occasionner entre les parties du vrai marbre. Cependant il désire, 1°. que la quantité de ce faux marbre qui remplit les vides laissés par le vrai marbre, soit *la plus petite* possible ; 2°. qu'en même temps l'étendue des deux cubes soit *la plus grande* possible. Quelle doit être la quantité de faux marbre ou de vide ? et quelles doivent être les dimensions des deux cubes ?

73. Puisque la quantité de faux marbre ou de vide doit être un *minimum*, l'addition faite au côté du cube composé de vrai et de faux marbre doit être aussi un *minimum*. De plus, puisque cette addition est étrangère au vrai marbre, elle doit être étrangère à ses dimensions, et par conséquent à leur longueur. Donc la ligne qui exprime une quelconque des dimensions du faux marbre doit être perpendiculaire à celle qui exprime la dimension correspondante du vrai marbre. Donc le côté du cube formé de vrai et de faux marbre doit être une ligne brisée.

Soit cette ligne brisée $(x \pm y\sqrt{-1})$, x étant la partie correspondante au vrai marbre, et $\pm y\sqrt{-1}$ la partie correspondante au faux. Nos cubes seront donc $(x \pm y\sqrt{-1})^3$ et $(x + 2 \pm y\sqrt{-1})^3$.

Puisque $y\sqrt{-1}$ doit être un *minimum*, sa variation doit être $= 0$. Puisque x est constant, sa variation doit aussi être $= 0$. Mais il y a ici une remarque à faire.

Si l'on prenoit les variations ou les différentielles de $(x \pm y\sqrt{-1})^3$ et de $(x + 2 \pm y\sqrt{-1})^3$ à l'ordinaire, on regarderoit par là même x et $\pm y\sqrt{-1}$ comme ayant leurs variations distinctes. On prendroit $(x \pm y\sqrt{-1})$, non comme une *seule* ligne brisée, mais comme *deux* lignes distinctes. Le faux marbre n'auroit d'autre *minimum* que zéro, et la question ne seroit pas résolue. Il faut donc lier $\pm y\sqrt{-1}$ à x. Pour cela, il faut les regarder comme le sinus et le cosinus d'un même arc et que cet arc seul varie.

74. Soit u l'arc dont y est le sinus et x le cosinus. Soit de plus e le nombre dont le logarithme hyperbolique est 1. On aura

$$(x \pm y\sqrt{-1})^3 = e^{l(x \pm y\sqrt{-1})^3} = e^{3l(x \pm y\sqrt{-1})} = e^{3l(\cos. u \pm \sin. u\sqrt{-1})}$$

$$= e^{3l\left\{\frac{\cos. u \pm \sin. u\sqrt{-1}}{\sqrt{\cos.^2 u + \sin.^2 u}}\right\} \times \sqrt{\cos.^2 u + \sin.^2 u}} = e^{3\left\{l\sqrt{\cos.^2 u + \sin.^2 u} + l\left(\frac{\cos. u \pm \sin. u\sqrt{-1}}{\sqrt{\cos.^2 u + \sin.^2 u}}\right)\right\}}$$

$$= \left\{\text{en remarquant que } l\frac{\cos. u \pm \sin. u\sqrt{-1}}{\sqrt{\cos.^2 u + \sin.^2 u}} = l\frac{\cos. u \pm \sin. u\sqrt{-1}}{\text{rayon}} = \pm u\sqrt{-1}\right\}$$

$$= e^{3\left\{l\sqrt{x^2+y^2} \pm u\sqrt{-1}\right\}} = e^{3\left\{l(xx+yy)^{\frac{1}{2}} \pm u\sqrt{-1}\right\}}, \text{ dont la différentielle est } \pm 3du\sqrt{-1} \times e^{3l(xx+yy)^{\frac{1}{2}} \pm 3u\sqrt{-1}}, \text{ ou } \pm 3du\sqrt{-1}\,(x \pm y\sqrt{-1})^3.$$

On a donc $\pm 3du\sqrt{-1}\ (x \pm y\sqrt{-1})^3 \pm 3du'\sqrt{-1}$

$:(x+2\pm y\sqrt{-1})^2=0$, en désignant par u l'arc dont $\pm y\sqrt{-1}$ est le sinus et $\overline{x+2}$ le cosinus. L'intégrale de cette équation est $(x\pm y\sqrt{-1})^3+(x+2\pm y\sqrt{-1})^2+C=0$. Pour déterminer la constante C, je remarque que, quand $y=0$, on doit avoir l'équation (7). Donc $C=-28$, et l'équation du *minimum* est $(x\pm y\sqrt{-1})^3+(x\pm y\sqrt{-1}+2)^2-28=0$.

Cette équation donne pour $x+y\sqrt{-1}$ les mêmes valeurs que l'équation (7.) pour x. On a donc - - -

$x\pm y\sqrt{-1}=1$, ou $=1+0\times\sqrt{-1}$; - - - (8).

$x\pm y\sqrt{-1}=-2+\sqrt{-6}$; - - - - (9).

$x\pm y\sqrt{-1}=-2-\sqrt{-6}$; - - - - (10).

75. Ces valeurs de $(x\pm y\sqrt{-1})$ sont des *minima* et non des *maxima*.

En effet l'équation (10), par exemple, c'est-à-dire, $x\pm y\sqrt{-1}+2+\sqrt{-6}=0$ est la racine d'une équation du second dégré dont l'autre racine est $x\mp y\sqrt{-1}+2-\sqrt{-6}=0$. Cette équation est donc $(x\pm y\sqrt{-1})(x\mp y\sqrt{-1})=$ $=(2+\sqrt{-6})(+2-\sqrt{-6})=+10$. - - -

Faisons $(x\pm y\sqrt{-1})=z$, et $(x\mp y\sqrt{-1})=+v$. Nous aurons $zv=10$, équation à l'hyperbole entre ses assymptotes. Or l'hyperbole est toute convexe par rapport à ses assymptotes. Si donc une de ses ordonnées ou de ses abscisses est ou un *maximum* ou un *minimum*, ce ne peut être qu'un *minimum*. En effet les diagonales des ligne brisées $(x\pm y\sqrt{-1})$ et $(x\mp y\sqrt{-1})$ sont des *minima* par rapport à cette hyperbole.

76. De plus, il est facile de montrer, d'après une autre considération, que la quantité du faux marbre est un *minimum*.

En effet, le vrai et le faux marbre sont liés de manière que leur ensemble forme une seule unité. Par conséquent les quantités de l'un et de l'autre sont en même temps ou des *minima* ou des *maxima*. Or la quantité du vrai marbre est zéro et par conséquent un *minimum* dans l'un des cubes, et la quantité du faux marbre, dans l'autre cube, est aussi un *minimum*; car on ne peut diminuer cette quantité de faux marbre sans diminuer en même temps les 28 cubes du vrai marbre, ou sans altérer la forme cubique.

77. J'ajoute que, si l'on construit les deux cubes comme je vais le dire, on trouvera que leur étendue est un *maximum*, parcequ'on ne pourroit l'augmenter sans opérer une solution de continuité, ou sans que la quantité du faux marbre ne cessât d'être un *minimum*.

D'ailleurs l'équation $(x\pm y\sqrt{-1})^3+(x+2\pm y\sqrt{-1})^3-28=0$, qui exprime l'étendue des cubes demandés et qui donne un *minimum* pour les côtés $(x\pm y\sqrt{-1})$ et $(x+2\pm y\sqrt{-1})$ de ces cubes, donne au contraire un *maximum* pour leur étendue. En effet elle se réduit à celle-ci: - - - - -

$(x\pm y\sqrt{-1}-1)\left\{(x\pm y\sqrt{-1})^2+4(x\pm y\sqrt{-1})+10\right\}=0$, ou

$\left\{x\pm y\sqrt{-1}+2+\sqrt{-6}\right\}\times\left\{x\pm y\sqrt{-1}+2-\sqrt{-6}\right\}=0$, ou

(à cause de l'indépendance des signes + et −, et de ce qu'ils ne marquent ici que des directions qui n'affectent point les quantités) - - - - - - - - -

$\left\{x\pm y\sqrt{-1}+2+\sqrt{-6}\right\}\times\left\{x\mp y\sqrt{-1}+2-\sqrt{-6}\right\}=0$.

Cette dernière équation donne - - - -

1°. $(x\pm y\sqrt{-1})(x\mp y\sqrt{-1})=10$, équation à l'hyperbole; (11) c'est celle du No. 75;

2°. $x^2+y^2=10$, équation au cercle - - - (12).

Dans l'équation (11), les coordonnées brisées qui expriment les côtés des cubes, ont (comme je l'ai dit) pour diagonale, un axe d'hyperbole qui est un *minimum*.

Dans l'équation (12), le premier membre qui quoique de deux dimensions seulement, exprime une nouvelle unité qui change l'étendue 28 du vrai marbre en une autre étendue plus considerable, ce premier membre, dis-je, représente le carré du rayon d'un cercle, lequel rayon est un *maximum*.

78. Ainsi les deux conditions de la question sont exactement remplies, quoiqu'elles semblent contradictoires. On voit par là que les imaginaires renfermées dans l'équation du problème, bien loin de confirmer cette contradiction, fournissent les moyens de concilier ces conditions.

79. Il ne s'agit plus que de construire les cubes donnés par les valeurs de $x \pm y\sqrt{-1}$ que présentent les équations (8), (9) et (10). Le premier qui est le seul appellé *réel* est spécialement exclu par l'état de la question. Construisons donc le second et le 3e.

Comme le faux marbre est destiné à remplir les vides laissés par le vrai marbre, il ne s'agit que de voir quels seront les vides contenus dans les cubes $(-2+\sqrt{-6})^3$ et $(-2-\sqrt{-6})^3$. Ces cubes sont - - - - - $-8+12\sqrt{-6}+36-6\sqrt{-6}$, et - - - - $-8-12\sqrt{-6}+36-6\sqrt{-6}$.

Or ces cubes renferment deux sortes de vides, savoir, 1°. ceux qu'indique le signe —; 2°. ceux qu'indique le signe $\sqrt{-1}$. Les premiers sont des *pleins soustractifs*. Les seconds sont des *vides absolus*.

80. En effet 1°. -8 est un cube qui a ses trois dimensions

et qui par conséquent est *plein*, et comme il porte le signe —, il est *soustractif*. 2°. Au contraire $-6\sqrt{-6}$ n'a que deux dimensions pleines.

Pour le prouver, je remarque que $\pm 6\sqrt{-6} = +\sqrt{6}.-\sqrt{6}.\mp\sqrt{-1}\sqrt{6}$. Or le signe $\sqrt{-1}$ attaché à $\sqrt{6}$ marque (No. 10.) que $\sqrt{-1}\sqrt{6}$ est perpendiculaire à ce qu'il seroit, s'il ne portoit pas ce signe; et s'il ne portoit pas ce signe, on auroit $+\sqrt{6}.-\sqrt{6}.\mp\sqrt{6} = \pm(\sqrt{6})^3$ qui est un cube dont les trois dimensions sont pleines. Puisque les trois dimensions de $\pm(\sqrt{6})^3$ sont pleines, les trois lignes marquées par $+\sqrt{6}$, $-\sqrt{6}$ et $\pm\sqrt{6}$ sont perpendiculaires entr'elles et la ligne marquée par $\pm\sqrt{6}$ est perpendiculaire au plan des deux lignes marquées, par $+\sqrt{6}$ et $-\sqrt{6}$. Si donc $\pm\sqrt{6}$ devient $\pm\sqrt{-1}.\sqrt{6}$, la ligne marquée par $\pm\sqrt{-1}.\sqrt{6}$ se trouvera nécessairement dans le plan des deux autres lignes. Le cube se trouvera donc réduit à un plan. Il ne sera donc plein que dans deux de ses dimensions. La troisième dimension sera donc ou vide ou nulle.

81. Maintenant je dis qu'elle sera vide et non pas nulle, ou plutôt, qu'elle sera *vide géométriquement et nulle arithmétiquement*.

Comme cette distinction est un point fondamental dans les principes que j'expose, il faut que je l'explique.

Si, après avoir parcouru une toise dans un sens quelconque, je la parcours une seconde fois en revenant au point de départ, le nombre des toises que je parcourrai sera $=2$, et la quantité dont je m'éloignerai du point de départ sera $=0$. Ces deux résultats donnent les deux significations de $+\sqrt{1-1}$. Le

premier donne sa signification *géométrique*, et le second, sa signification *arithmétique*. Le premier exprime les opérations, et le second en exprime la conséquence. Le premier représente les mesures prises qui sont les données du calcul *à faire*, et le second représente le calcul *fait* d'après ces données. Pour savoir donc si ce sont les données ou leur conséquence qu'exprime $+\sqrt{6}$, $-\sqrt{6}$, $+\sqrt{-1}\sqrt{6}$, $-\sqrt{-1}\sqrt{6}$, il faut examiner ce qui est demandé par la question. Or, la question est de construire un cube et de déterminer ce qu'il contient de matière réelle. Mais pour construire un cube, il faut avoir son côté et l'étendre dans les trois dimensions. Pour déterminer ce qu'il contient de matière réelle (lorsqu'on sait d'ailleurs qu'il en peut contenir qui ne le soit pas) il faut savoir combien de matière réelle s'étend dans ses trois dimensions. Le premier point de la question, qui consiste à *mesurer*, est géométrique. Le second, qui consiste à *compter*, est arithmétique. Or la réponse au premier point de la question donne *trois* dimensions, et celle au second point n'en donne que *deux*. La troisième dimension est donné *réelle géométriquement et nulle arithmétiquement*. Cependant elle ne peut pas être réelle et nulle en même temps. Elle renferme donc deux choses dont l'une est réelle et l'autre nulle. C'est donc un vide et non une nullité absolue. La matière est nulle, mais l'espace est réel. La 3e. dimension est donc *vide géométriquement et nulle arithmétiquement*. Donc le signe $\sqrt{-1}$ mis devant l'expression d'un cube ou d'un parallélipipède marque *un vide* dont l'étendue est égale à celle de ce cube ou de ce parallélipipède.

82. Il s'agit maintenant d'expliquer la différence qui se

trouve entre ce que j'ai appelé *plein soustractif* et ce que j'ai nommé *vide absolu*.* Le plein soustractif est, comme je l'ai dit, désigné par le signe —, et le vide absolu par le signe $\sqrt{-1}$. La différence entre le plein soustracteur et le vide absolu n'est donc que la différence de ce qu'expriment les signes — et $\sqrt{-1}$. Le signe $\sqrt{-1}$ applique à un espace qui s'étend dans les trois dimensions, marque *par lui-même* la destruction, non pas de l'espace qui a été rempli, mais de la matière qui le remplissoit. Le signe — marque cette même destruction, mais non *par lui-même*. Il faut, pour qu'il la marque, qu'il soit accompagné du signe +. Ainsi, par exemple, dans le cube $-8-12\sqrt{-6}+8+6\sqrt{-6}$, pour que -8 représente un vide, il faut écrire ce cube ainsi : $-8-12\sqrt{-6}+8+28+6\sqrt{-6}$. Or je dis que $-8+8$ désigne un vide dont la partie -8 s'étend selon la direction marquée par —, et la partie $+8$ s'étend selon la direction marquée par +. Conséquemment, l'espace $-8+8+28$ considéré *géométriquement*, c'est-à-dire, par rapport à son *étendue abstraite*, renferme les trois espaces désignés par 28, -8 et $+8$, savoir, 28 pieds cubiques dans le sens +, 8 cubiques dans le sens — et 8 pieds cubiques dans le sens —. Il renferme donc 44 pieds cubiques. Mais ce même espace considéré *arithmétiquement*, c'est-à-dire, par rapport à son *étendue matérielle* ne contient que l'espace marqué par la valeur

* (N°.) Ces considérations sur le plein et le vide peuvent paroître plus que singulières dans une question de géométrie ; mais puisque j'introduis dans mon analyse des signes qui ont deux significations indépendantes l'une de l'autre, il faut bien que j'obtienne des résultats de deux espèces, c'est-à-dire, des résultats dont l'un ait un caractère que l'autre n'a pas. *La présence* et l'*absence* de ce caractère sont ce que j'appelle *plein* et *vide*, quel que soit ce caractère.

arithmétique de $-8+8+8$ qui est $=8$. Ainsi, tandis que $+6\sqrt{6}\sqrt{-1}$ représente un vide cubique $=6\sqrt{6}$, -8 conduit à un vide $=8\times a$.

83. Ce résultat, tout paradoxal qu'il est, est la conséquence d'une propriété remarquable des quantités imaginaires, savoir, d'être des *logarithmes*.

N°. Qu'il me soit permis ici d'interrompre mon sujet, pour m'arrêter un peu sur les propriétés logarithmiques du signe $\sqrt{-1}$.

84. [*Propriétés logarithmiques du signe* $\sqrt{-1}$. Je dis qu'en général, si l'on regarde $\sqrt{-1}$, non comme la racine (racine impossible) du *carré arithmétique* -1 (carré pareillement impossible), mais comme le signe de l'*opération géométrique* par laquelle on élève une perpendiculaire, on aura $(\sqrt{-1})^n=\pm n(\sqrt{-1})$.

Pour le prouver, supposons d'abord n un nombre entier impair. Nous aurons $\sqrt{-1}=e^{\frac{\pi}{2}\sqrt{-1}}$, $(\sqrt{-1})^2=e^{\pi\sqrt{-1}}$, $(\sqrt{-1})^3=e^{(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2})\sqrt{-1}}$, &c. Donc $(\sqrt{-1})^n=e^{(90^\circ+90^\circ+90^\circ+\&c.)\sqrt{-1}}$.

Mais que peut signifier cet exposant $(90^\circ+90^\circ+\&c.)\sqrt{-1}$? Dans ax^n, l'exposant n marque que x multiplié a autant de fois que n renferme d'unités. Ainsi, dans ax^n, n est un signe d'opérations arithmétiques. Mais aucune opération arithmétique ne peut être désignée par $(90^\circ+90^\circ+\&c.)\sqrt{-1}$. Cet exposant est donc le signe d'opérations *purement géométriques* (No. 2), c'est-à-dire, d'opérations où l'on n'a en vue que les *directions*, sans considérer les *longueurs*.

85. Il n'y a nulle analogie entre une *multiplication* et une *direction*. Comment donc se faire une idée claire d'un exposant qui indique *une somme de directions*, quand l'idée qu'on a coutume d'attacher exclusivement à un exposant est, qu'il indique *une somme de multiplications*?

Un seul moyen se présente. C'est de trouver une idée complexe qui renferme les deux idées de *multiplication* et de *direction*. Or *la mesure de l'étendue* présente cette idée complexe (No. 2).

Prenons pour exemple le carré ABCD (Fig. 11). Pour mesurer ce carré, *je porte la mesure* de A en B, puis de B en D, dans une *direction perpendiculaire* à celle de AB. Voilà l'idée de *direction*. Ensuite *je compte* séparément les parties de AB et les parties de BC, puis je *multiplie* le nombre des premières par celui des secondes. Voilà l'idée de *multiplication*. De ces deux idées ne conservons que la première, puisque la seconde est exclue par le signe $\sqrt{-1}$. Dans ce cas, l'exposant $(90^\circ + 90^\circ + \&c.)\sqrt{-1}$ exprimera un simple arc de cercle *perpendiculaire au rayon*, et le nombre des multiplications qu'il exprimera sera $= 0$. De là il suit que, pour ramener l'exposant $(90^\circ + 90^\circ + \&c.)\sqrt{-1}$ aux exposans ordinaires, c'est-à-dire, aux exposans qui sont des signes de multiplications, il faut le multiplier par 0. On aura donc $e^{0\times(90^\circ+90^\circ+\&c.)\sqrt{-1}} = e^{0\times\pi\,.\,90^\circ\sqrt{-1}} = 1 + 0\times\pi\,.\,90^\circ\sqrt{-1} = 1 + 0\times\pi\,.\,90^\circ\sqrt{-1}\,.\,le$ =(voyés la note cy dessous)* $1 + 0\times\pi\,.\,90^\circ\sqrt{-1}$.

* On peut objecter contre cette équation que le 0 que j'ai ajouté au terme $\pi\,.\,90^\circ\sqrt{-1}$ ne le rend point nul, et que par conséquent la véritable équation n'est pas $e^{0\times\pi\,.\,90^\circ\sqrt{-1}} = 1 + 0\times\pi\,.\,90^\circ\sqrt{-1}$, mais

L'expression $0 \times n \cdot 90^\circ\sqrt{-1}$ renferme deux parties, savoir, $n \cdot 90^\circ\sqrt{-1}$ et 0.

Soit (Fig. 11.) l'arc $AEI = n \cdot 90^\circ\sqrt{-1}$. Le signe $\sqrt{-1}$ que renferme $n \cdot 90^\circ\sqrt{-1}$ marque que $\overset{\frown}{AEI}$ est perpendiculaire au plan de ce papier. En effet si $n \cdot 90^\circ$ ne portoit pas le signe $\sqrt{-1}$, il seroit déja perpendiculaire au rayon $\overline{CA}$. Or le signe $\sqrt{-1}$ ne peut pas le faire cesser d'être perpendiculaire à ce rayon, puisqu'il ne peut pas le faire cesser d'être un arc de cercle décrit de ce rayon. Donc le signe $\sqrt{-1}$

$$e^{0\times n\cdot 90^\circ\sqrt{-1}} = 1 + 0\times n\cdot 90^\circ\sqrt{-1} + \frac{(0\times n\cdot 90^\circ\sqrt{-1})^2}{1\cdot 2} + \frac{(0\times n\cdot 90^\circ\sqrt{-1})^3}{1\cdot 2\cdot 3} + \&c.$$

Pour prévenir cette objection, je remarque

1°. Que ce qui est représenté par une série ne peut être que le résultat d'une suite d'opérations purement arithmétiques; que par conséquent on n'y doit considérer que la valeur arithmétique. Or $0\times n \cdot 90^\circ\sqrt{-1}$ est *arithmétiquement*

$$= 0\times n\cdot 90^\circ\sqrt{-1} + \frac{(0\times n\cdot 90^\circ\sqrt{-1})}{1\cdot 2} + \&c.$$

C'est ce qu'il est facile de montrer. En effet si $n \cdot 90^\circ\sqrt{-1}$ exprime quelque chose, ce ne peut être qu'un arc indéfini $\overset{\frown}{AEI}$ (Fig. 11.) qui (comme je le prouverai dans le texte) doit être perpendiculaire au plan de ce papier. Cet arc est purement descriptif. La chose dont il détermine la position est la seule qu'on doive considérer dans la série précédente. Mais quelle est cette chose dont l'arc $\overset{\frown}{AEI}$ détermine la position? Cette chose, quelle qu'elle soit, doit être sur le plan de ce papier. Or l'arc $\overset{\frown}{AEI}$ étant perpendiculaire au plan de ce papier ne peut déterminer sur ce plan que deux points. La chose qu'il y détermine se réduit donc à deux points dont la *valeur arithmétique* est nulle. Donc tous les termes de la série précédente, à l'exception du premier (qui est 1) sont arithmétiquement nuls.

Je remarque 2°. que les termes qui, dans cette série suivent le terme $0\times n \cdot 90^\circ\sqrt{-1}$ n'ajoutent rien à sa *valeur descriptive*. En effet tous ces termes sont rendus descriptifs par *le signe descriptif* $n \cdot 90^\circ\sqrt{-1}$ qu'ils contiennent. Chacun de ces termes se réduit donc au signe descriptif $n \cdot 90^\circ\sqrt{-1}$ répété un nombre quelconque de fois. Cette répétition modifie à la vérité l'étendue de l'arc $\overset{\frown}{AEI}$, mais elle ne lui

joint au signe de l'arc de cercle $n \cdot 90°$ marque une double perpendicularité, c'est-à-dire, la perpendicularité à un plan. Or ce plan ne peut être que celui de ce papier. A l'égard de o, nous l'avons posé pour exprimer une *nullité de multiplication*. $o \times m \times 90°\sqrt{-1}$ n'exprime donc qu'*une trace de ligne* mise au bout de la ligne $\overline{CA}$ et qui lui est perpendiculaire. Donc, en représentant $o \times n \cdot 90°\sqrt{-1}$ par $T(\widehat{AEI})$ qui signifie *trace de* $\widehat{AEI}$, nous aurons (supposé $\overline{CA} = 1$) $1 + o \times n \cdot 90°\sqrt{-1} = \overline{CA} + T(\widehat{AEI})$. Donc

$$\frac{e^{+T(\widehat{AEI})} - e^{-T(\widehat{AEI})}}{2} = \frac{\left\{\overline{CA} + T(\widehat{AEI})\right\}}{2} - \frac{\left\{\overline{CA} - T(\widehat{AEI})\right\}}{2} = T(\widehat{AEI})$$

86. Ainsi, quoiqu'on ait $\frac{e^{+\widehat{AEI}\sqrt{-1}} - e^{-\widehat{AEI}\sqrt{-1}}}{2\sqrt{-1}} = \sin.\widehat{AEI}$, on a $\frac{e^{+T(\widehat{AEI})} - e^{-T(\widehat{AEI})}}{2} = T(\widehat{AEI})$. Pour que ce point paroisse

fait pas décrire d'autres points sur le plan de ce papier. Elle n'ajoute donc rien à sa valeur descriptive.

Je remarque 3°. que la série précédente est $= \cos. o \times n \cdot 90° \pm \sin. o \times n \cdot 90°\sqrt{-1} = 1$. Or cette même série $= 1 +$ une suite de termes dont $o \times n \cdot 90°\sqrt{-1}$ est un des facteurs. Désignons cette suite de termes par T. Puisque $o \times n \cdot 90°\sqrt{-1}$ est un des facteurs de T, T est donc (arithmétiquement) $= o$. Il est donc purement descriptif. Or le rayon 1 exprime la distance du cercle et la circonférence. Donc la série entière $(1 + T)$ traduite en langage descriptif signifie :

la distance 1 plus la description de l'arc $\widehat{AEI}$, ou, en d'autres termes,

la trace de l'arc $\widehat{AEI}$ *décrit à la distance 1 d'un point pris comme centre.*

Or $e^{o \times n \cdot 90°\sqrt{-1}}$ et $1 + o \times n \cdot 90°\sqrt{-1}$ ont l'un et l'autre cette même signification. Donc l'équation $e^{o \times n \cdot 90°\sqrt{-1}} = 1 + o \times n \cdot 90°\sqrt{-1}$ est une équation identique.

clair, il suffit de remarquer, 1°. que ce qui est exprimé par $\widehat{AEI}$ renferme deux idées, savoir celle de *nombre* et celle de *direction*; 2°. que lorsqu'on détermine un *sinus*, c'est sa *valeur numérique* seule qu'on détermine; 3°. que le signe $\sqrt{-1}$ exclut toute idée de nombre aussi bien que le signe o; 4°. qu'il en est de même de ce que je me suis proposé de désigner par $T(\widehat{AEI})$; 5°. que $\frac{e^{+\widehat{AEI}\sqrt{-1}} - e^{-\widehat{AEI}\sqrt{-1}}}{2\sqrt{-1}}$, en anéantissant le signe $\sqrt{-1}$ de l'exposant par le signe $\sqrt{-1}$ du dénominateur, et en n'admettant pas le signe o, ne laisse que la valeur *numérique*; 6°. qu'au contraire $\frac{e^{+T(\widehat{AEI})} - e^{-T(\widehat{AEI})}}{-2}$, en n'anéantissant pas le signe $\sqrt{-1}$ qui exclut toute idée de nombre, et en admettant le signe o, ne laisse subsister que la valeur *descriptive*.

87. On peut donc poser pour principe général, que $\frac{e^{+nx\sqrt{-1}} - e^{-nx\sqrt{-1}}}{2}$ *divisé par* $\sqrt{-1}$ *est* $= \sin. nx$, *tandis que* $\frac{e^{+ox\sqrt{-1}} - e^{-ox\sqrt{-1}}}{2}$ *non divisé par* $\sqrt{-1}$ *et* $= n \cdot T(x)$.

Or je n'ai mis le signe o que pour ôter, par un signe connu, toute idée de multiplication; mais le signe $\sqrt{-1}$ seul suffit pour exclure cette idée. On peut donc supprimer le zéro.

Donc $\frac{e^{+nx\sqrt{-1}} - e^{-nx\sqrt{-1}}}{2} = nT(x)$. Donc $(\sqrt{-1})^n =$

$= \frac{e^{\frac{n\pi}{2}\sqrt{-1}} - e^{-\frac{n\pi}{2}\sqrt{-1}}}{2}$ (n étant un nombre entier impair) $=$

$= n \cdot T\left(\frac{\pi}{2}\right) = n(\sqrt{-1})$.

88. Cette équation, toute singulière qu'elle peut paroître, n'est que l'expression algébrique du principe posé au commencement de ce mémoire, savoir, *que* $\sqrt{-1}$ *est le signe de la perpendicularité, abstraction faite de toute longueur de ligne.* En effet le premier membre $(\sqrt{-1})^n$ marque que *le signe* $\sqrt{-1}$ *de la perpendicularité* est répeté n fois, c'est-à-dire, est attaché n fois à la même ligne 1. Le 2d. membre $n \cdot T\left(\frac{\pi}{2}\right)$ marque que l'arc $\frac{\pi}{2}$ qui est *la mesure de la perpendicularité* est tracé n fois.

89. J'ai supposé n un nombre entier impair, afin que le signe $\sqrt{-1}$ ne s'anéantit pas. Si n étoit un nombre entier pair, le signe $\sqrt{-1}$ s'anéantiroit, et l'on pourroit croire qu'alors l'équation $(\sqrt{-1})^n = n \cdot T\left(\frac{\pi}{2}\right)$ n'auroit pas lieu, puisque (No. 86) j'ai fondé la vérité de cette équation sur ce que le signe $\sqrt{-1}$ qu'elle renferme exclut toute idée de nombre. Mais cette difficulté n'est qu'apparente. En effet lorsque n est un nombre entier pair, ce n'est pas le signe $\sqrt{-1}$ qui disparoit, mais le sinus de l'arc $n \cdot \frac{\pi}{2}$ qui devient $= 0$. Alors, l'idée de nombre, bien loin de se trouver rétablie, se trouve doublement exclue, savoir, une fois par le signe $\sqrt{-1}$ qui subsiste toujours, et une seconde fois par l'anéantissement de la valeur numérique du sinus. Je dis que le signe $\sqrt{-1}$ subsiste toujours, parce qu'on ne peut l'anéantir qu'en le divisant par lui-même.

90. L'équation $(\sqrt{-1})^n = n \cdot T\left(\frac{\pi}{2}\right)$ est donc aussi vraie, lorsque n est un nombre entier positif. Maintenant je dis qu'elle est également vraie, quel que soit n, entier ou fractionnaire, positif ou négatif. Pour cela, il suffit de substituer

à l'idée de *perpendicularité* l'idée d'une *inclinaison* quelconque. Dans ce cas, les deux membres de l'équation $(\sqrt{-1})^{\pm\frac{n}{m}} = \pm\frac{n}{m}\,T\left(\frac{\pi}{2}\right)$ exprimeront l'inclinaison $\frac{\pi}{2m}$ répétée n fois dans le sens $+$ ou dans le sens $-$.

91. De ce qui vient d'être dit je tire deux conséquences.

Premiere conséquence. Soit R un rayon quelconque. $R(\sqrt{-1})^{\alpha}$ exprime la trace d'un arc quelconque $\frac{\alpha\pi}{2}$ décrit du rayon R. Si $R = \frac{r}{o}$ et $n = p \times o$, $\frac{r}{o}(\sqrt{-1})^{p\times o}$ exprime la trace d'une ligne droite quelconque. Par là *on peut écrire algébriquement et calculer à la manière des logarithmes, les traces d'un nombre quelconque de lignes avec leurs longueurs et leurs positions.* Je regrette de ne pouvoir m'étendre ici sur cette propriété du signe $\sqrt{-1}$ qui paroît devoir être d'une grande utilité dans la géométrie descriptive.

92. Seconde conséquence. *Pour passer du logarithme* $\pm\frac{n}{m}\,T\left(\frac{\sqrt{}}{2}\right)$ *à son exponentielle, il suffit de diviser ce logarithme par* $\sqrt{-1}$. *voyez le No.* 86.

93. Revenons à nos cubes de vrai et de faux marbre.

Ces cubes sont $= -8 \pm 12\sqrt{-6} + 36 \mp 6\sqrt{6}$. Pour avoir le faux marbre qu'ils contiennent, il faut les exprimer ainsi: $8(\sqrt{-1})^2 + 12\cdot\sqrt{6}(\sqrt{-1})^{\pm 1} + 36(\sqrt{-1})^2 + 6\sqrt{6}(\sqrt{-1})^{-1}$ qui donne (No. 84.) $(2\cdot 8 \pm 12\cdot\sqrt{6} + 0\cdot 36 - 6\sqrt{6})\sqrt{-1}$. Le signe $\sqrt{-1}$ marque ici que les signes $+$ et $-$ ne sont pas des signes d'addition et de soustraction. Par conséquent la quantité du faux marbre est $2\cdot 8 + 12\sqrt{6} + 6\sqrt{6} = 16 + 18\sqrt{6}$.

Pour avoir le vrai marbre que contiennent nos cubes, il faut les exprimer ainsi; $-8+12\sqrt{6}\left[\pm T\left(\frac{\pi}{2}\right)\right]+36+6\sqrt{6}\left[-T\left(\frac{\pi}{2}\right)\right]$. Or (No. 86.) le signe $T\left(\frac{\pi}{2}\right)$ anéantit toute valeur numérique. Donc la quantité du vrai marbre se réduit à $-8+36=28$, comme le problème l'exige.

94. *Regle générale.* Etant donné une expression telle que $(A\pm B\sqrt{-1})^{\frac{n}{m}}$ dont le développement contient des termes affectés du signe $\sqrt{-1}$ et des termes qui n'en sont pas affectés, on connoîtra la somme des termes réellement affectés de ce signe, par la regle suivante : Développez $(A\pm B\sqrt{-1})^{\frac{n}{m}}$ suivant la formule du binome, en observant, 1°. de donner à $\sqrt{-1}$ les exposans indiqués par cette formule, 2°. que $(\sqrt{-1})^{4i}$ (i étant un nombre entier quelconque positif ou négatif) fait le même effet que 0; 3°. Qu'il ne faut faire aucune autre réduction que celle de $4i$ à 0. Cela fait, tous les termes qui ne seront pas affectés de $(\sqrt{-1})^{4i}$ resteront. Leur somme sera donc celle des quantités réellement affectées du signe $\sqrt{-1}$.

On voit que, dans ce cas, -1 se trouve être ce qu'on appelle un carré imaginaire.

95. Sur la solution du 5e. problème (No. 70), je remarque 2°. Que l'équation (7.) qui est du 3e. dégré n'est pas dans le cas irreductible. On peut cependant à l'aide des imaginaires, la résoudre comme celles qui sont dans ce cas. C'est ce qu'il faut montrer.

Cette équation développée donne celle ci: - - - -
$x^3+3x^2+6x-10=0$ - - - - - (13).
Je fais $x=y-1$, pour anéantir le second terme, ce qui me donne $y^3+3y-14=0$ - - - - - (14).

En imitant le procédé dont on fait usage dans le cas irréductible, si l'on représente par $a\sqrt{-1}$ l'angle dont le sinus est -14 et le rayon $2\sqrt{-1}$, on aura pour les trois racines de l'équation (14) $y=\sin.\frac{a\sqrt{-1}}{3}$, $y=\sin.\left\{\left(60^\circ-\frac{a}{3}\right)\right\}\sqrt{-1}$,
et $y=-\sin.\left\{\left(60^\circ+\frac{a}{3}\right)\sqrt{-1}\right\}$, - - - -
ou bien (en multipliant ces sinus par leur rayon $2\sqrt{-1}$, pour les ramener au rayon des tables que je suppose $=1$)
$y=2\sqrt{-1}\times\sin.\frac{a\sqrt{-1}}{3}$, $y=2\sqrt{-1}\times\sin.\left\{\left(60^\circ-\frac{a}{3}\right)\sqrt{-1}\right\}$,
et $y=-2\sqrt{-1}\times\sin.\left\{\left(60^\circ+\frac{a}{3}\right)\sqrt{-1}\right\}$.

On verra dans la question suivante (No. 105.) ce que c'est que ces sinus imaginaires plus grands que leur rayon. En attendant, il faut montrer analytiquement que les trois racines qui viennent d'être posées résolvent l'équation (14).

96. Pour donner à cette recherche toute la généralité possible, supposons que l'équation proposée soit $y^3+py+q=0$ (15), et que cette équation ne soit pas dans le cas irréductible, et même que p soit positif.

Je désigne par $a\sqrt{-1}$ l'angle dont le sinus est $\frac{3q}{p}$ (qui, pour l'équation (14), se réduit à -14) et dont le rayon est $2\sqrt{-\frac{p}{3}}$ (qui, pour l'équation (14), se reduit à $2\sqrt{-1}$); et j'ai pour l'équation (15), comme pour l'équation (14),

$$\begin{cases} y = +2\sqrt{-\frac{p}{3}} \times \sin.\frac{a\sqrt{-1}}{3}; \\ y = +2\sqrt{-\frac{p}{3}} \times \sin.\left\{(60° - \frac{a}{3})\sqrt{-1}\right\}; \\ y = -2\sqrt{-\frac{p}{3}} \times \sin.\left\{(60° + \frac{a}{3})\sqrt{-1}\right\}. \end{cases} \quad (16).$$

97. Pour montrer que ces trois racines sont les vraies racines de l'équation (15), il faut les développer.

$$1°.\ y = +2\sqrt{-\frac{p}{3}} \times \sin.\frac{a\sqrt{-1}}{3} = \sqrt{-\frac{p}{3}}\left\{(\times \sin.\frac{a\sqrt{-1}}{3} + \cos.\frac{a\sqrt{-1}}{3}) + (\sin.\frac{a\sqrt{-1}}{3} - \cos.\frac{a\sqrt{-1}}{3})\right\}.$$

Or supposons que le plan sur lequel l'arc $\frac{a}{3}$ est décrit soit perpendiculaire à celui de ce papier. Si le sinus de cet arc est parallèle au plan de ce papier (comme on est maître de le supposer,) son cosinus sera perpendiculaire à ce même plan. Alors $\sqrt{-\frac{p}{3}} \times \sin.\frac{a\sqrt{-1}}{3}$ restera comme il est, parceque les deux signes de perpendicularité que ce terme renferme se détruisent l'un l'autre; $\sqrt{-\frac{p}{3}} \times \cos.\frac{a\sqrt{-1}}{3}$, au contraire, se changera en $\sqrt{-\frac{p}{3}} \times \cos.\frac{a}{3}$. Donc

$y = +\sqrt{-\frac{p}{3}}\left\{(\sin.\frac{a\sqrt{-1}}{3} + \cos.\frac{a\sqrt{-1}}{3}) + (\sin.\frac{a\sqrt{-1}}{3} - \cos.\frac{a\sqrt{-1}}{3})\right\}$ devient

$y = +\sqrt{-\frac{p}{3}}\left\{(\sin.\frac{a\sqrt{-1}}{3} + \cos.\frac{a}{3}) + (\sin.\frac{a\sqrt{-1}}{3} - \cos.\frac{a}{3}\right) = -$

$= +\sqrt{-\frac{p}{3}}\left\{(\sin.a\sqrt{-1}\cos.a)^{\frac{1}{3}} + (\sin.\frac{a\sqrt{-1}}{3} - \cos.\frac{a}{3})^{\frac{1}{3}}\right\} = -$

$= +\sqrt{-\frac{p}{3}}\left\{\left(\sin.(a\sqrt{-1}) + \sqrt{\text{rayon}^2 - (\sin.(a\sqrt{-1}))^2}\right)^{\frac{1}{3}} + \left(\sin.(a\sqrt{-1}\right.\right.$

$\left.\left. - \sqrt{\text{rayon}^2 - (\sin.(a\sqrt{-1}))^2}\right)^{\frac{1}{3}}\right\} =$

$= +\sqrt{-\frac{p}{3}}\left\{(\sin.a\sqrt{-1} + \sqrt{1+\sin.^2 a})^{\frac{1}{3}} + (\sin.a\sqrt{-1} - \sqrt{1+\sin.^2 a})^{\frac{1}{3}}\right\} =$

$$= \left[\sin.(a\sqrt{-1}) \times \sqrt{\left(-\frac{p}{3}\right)^3} + \sqrt{\left(-\frac{p}{3}\right)^6 + \sin.^2 a\left(-\frac{p}{3}\right)^3}\right]^{\frac{1}{3}}$$

$$+ \left[\sin.(a\sqrt{-1}) \times \sqrt{\left(-\frac{p}{3}\right)^3} - \sqrt{\left(-\frac{p}{3}\right)^6 + \sin.^2 a\left(-\frac{p}{3}\right)^3}\right]^{\frac{1}{3}}.$$

98. Puisque $a\sqrt{-1}$ est l'angle dont le sinus est $\frac{M}{p}$ et le rayon 2 $\sqrt{-\frac{p}{3}}$, on a pour le rayon 1, $+\frac{M}{p} = -\frac{q}{2}$

$$+\sin.a\sqrt{-1} = \frac{-\frac{q}{2}}{2\sqrt{-\frac{p}{3}}} = \frac{-\frac{q}{2}}{\left(-\frac{p}{3}\right)^{\frac{3}{2}}}.$$ De plus $\sin.(a\sqrt{-1})$

$= (\sin. a)\sqrt{-1}$. Par conséquent $\sin.^2(a\sqrt{-1}) = -\sin.^2 a$.

Ainsi $\sin.^2 a = -\sin.^2(a\sqrt{-1}) = -\frac{\left(-\frac{q}{2}\right)^2}{\left(-\frac{p}{3}\right)^3} = \frac{\left(\frac{q}{2}\right)^2}{\left(\frac{p}{3}\right)^3}$. La valeur précedente de y devient donc

$$y = \left[-\frac{q}{2} + \sqrt{-\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}\right]^{\frac{1}{3}} + \left[-\frac{q}{2} - \sqrt{-\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}\right]^{\frac{1}{3}}.$$

99. Mais cette valeur de y tire son origine d'un arc perpendiculaire au plan de ce papier, aussi bien que son cosinus qui est $\sqrt{-\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2}$ ou $\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} \times \sqrt{-1}$. Si l'on remet cet arc sur le plan de ce papier, son cosinus s'y trouvera et son sinus y restera. Le sinus conservera donc son signe, mais le cosinus perdra le sien qui est $\sqrt{-1}$. On aura donc en dernière analyse

$$y = \left[-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}\right]^{\frac{1}{3}} + \left[-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}\right]^{\frac{1}{3}} \quad (17)$$

C'est la formule connue sous le nom de formule de Cardan.

100. 2°. $y = 2\sqrt{-\frac{p}{3}} \times \sin.\left\{(60° - \frac{a}{3})\sqrt{-1}\right\} =$

$$= \sqrt{-\tfrac{p}{3}}\left[\sin.\left\{\left(60^\circ - \tfrac{a}{3}\right)\sqrt{-1}\right\} + \cos.\left(60^\circ - \tfrac{a}{3}\right) + \right.$$
$$\left. + \sin.\left\{\left(60^\circ - \tfrac{a}{3}\right)\sqrt{-1}\right\} - \cos.\left(60^\circ - \tfrac{a}{3}\right)\right] =$$
$$= \sqrt{-\tfrac{p}{3}}\left[\left\{\sin.\left(60^\circ - \tfrac{a}{3}\right)\right\}\sqrt{-1} + \cos.\left(60^\circ - \tfrac{a}{3}\right) + \right.$$
$$\left. + \left\{\sin.\left(60^\circ - \tfrac{a}{3}\right)\right\}\sqrt{-1} - \cos.\left(60^\circ - \tfrac{a}{3}\right)\right] =$$
$$= \sqrt{-\tfrac{p}{3}}\left[\left(\cos.60^\circ + \sin.60^\circ\sqrt{-1}\right)\left(\cos.\tfrac{a}{3} - \sin.\tfrac{a}{3}\sqrt{-1}\right)\right.$$
$$\left. - \left(\cos.60^\circ - \sin.60^\circ\sqrt{-1}\right)\left(\cos.\tfrac{a}{3} + \sin.\tfrac{a}{3}\sqrt{-1}\right)\right] =$$
$$= \sqrt{-\tfrac{p}{3}}\left[\left(\cos.60^\circ + \sin.60^\circ\sqrt{-1}\right)\left(\cos.a - \sin.a\sqrt{-1}\right)^{\frac{1}{3}}\right.$$
$$\left. - \left(\cos.60^\circ - \sin.60^\circ\sqrt{-1}\right)\left(\cos.a + \sin.a\sqrt{-1}\right)^{\frac{1}{3}}\right] =$$
$$= \left(\cos.60^\circ + \sin.60^\circ\sqrt{-1}\right) \times - \sqrt{-\tfrac{p}{3}}\left(\sin.a\sqrt{-1} - \cos.a\right)^{\frac{1}{3}}$$
$$+ \left(\cos.60^\circ - \sin.60^\circ\sqrt{-1}\right) \times - \sqrt{-\tfrac{p}{3}}\left(\sin.a\sqrt{-1} + \cos.a\right)^{\frac{1}{3}}.$$

Or $\cos.60^\circ = \frac{1}{2}$; $\sin.60^\circ = \frac{\sqrt{3}}{2}$; - - -

$$- \sqrt{-\tfrac{p}{3}}\left(\sin.a\sqrt{-1} \mp \cos.a\right)^{\frac{1}{3}} = (\text{Nos. } 97, 98, 99)\left[+\tfrac{q}{2} \pm \right.$$
$$\left. \pm \sqrt{\left(\tfrac{p}{3}\right)^3 + \left(\tfrac{q}{2}\right)^2}\right]^{\frac{1}{3}}. \quad \text{Donc } y = \left(\tfrac{1+\sqrt{-3}}{2}\right)\left(\tfrac{q}{2} + \right. \quad - \quad -$$
$$\left. + \sqrt{\left(\tfrac{p}{3}\right)^3 + \left(\tfrac{q}{2}\right)^2}\right]^{\frac{1}{3}} + \left(\tfrac{1-\sqrt{-3}}{2}\right)\left(\tfrac{q}{2} - \sqrt{\left(\tfrac{p}{3}\right)^3 + \left(\tfrac{q}{2}\right)^2}\right]^{\frac{1}{3}} \quad (18).$$

C'est la formule générale pour la seconde racine de l'équation (15).

101. 8°. Il est facile maintenant de voir de la 3e. racine de cette équation, laquelle racine est - - -

$y = 2\sqrt{-\frac{p}{3}} \times \sin.\left\{\left(60^\circ + \frac{a}{3}\right)\sqrt{-1}\right\}$, devient - -

$$y = \left(\tfrac{1-\sqrt{-3}}{2}\right)\left(\tfrac{q}{2} + \sqrt{\left(\tfrac{p}{3}\right)^3 + \left(\tfrac{q}{2}\right)^2}\right)^{\frac{1}{3}} + \left(\tfrac{1+\sqrt{-3}}{2}\right)\left(\tfrac{q}{2} - \right. \quad -$$

$-\sqrt{\left[\frac{r}{3}\right]^2+\left[\frac{x}{2}\right]^3}$ (19). C'est la formule générale pour la 3e. racine de l'équation (15).

102. On voit donc par ce détail que les formules (17), (18) et (19) qui sont les formules générales pour la résolution des équations du 3e. dégré, sont les mêmes que les formules (16). Mais celles-ci ne diffèrent des formules pour la résolution de ces mêmes équations, dans le cas irréductible, qu'en ce que l'arc et le rayon portent le signe $\sqrt{-1}$, c'est-à-dire, qu'ils sont dans un plan perpendiculaire à celui où ils seroient dans ce cas. J'ai donc prouvé ce que j'avois intention de prouver, savoir, qu'on peut résoudre les équations du 3e. dégré, dans tous les cas, de la même manière qu'on le fait dans le cas irréductible, c'est-à-dire, *par un sinus*.

103. Il ne me reste plus qu'un point à discuter sur cette matière; c'est de savoir ce que signifient des sinus ou des cosinus imaginaires plus grands que leurs rayons. C'est ce que je vais faire dans la question qui suit.

Problème VI.

104. Que deviennent les courbes du 2d. dégré, c'est-à-dire, les sections coniques, lorsque leurs ordonnées deviennent imaginaires?

Commençons par le cercle. Son équation est, en mettant au centre l'origine des coordonnées, $yy=aa-xx$. Lorsque $x>a$, on a $yy=-(xx-aa)$, et $y=\pm\sqrt{xx-aa}\,.\sqrt{-1}$, ou bien $y\sqrt{-1}=\mp\sqrt{xx-aa}$. Si l'équation $y=\pm\sqrt{aa-xx}$ exprime l'ordonnée d'un cercle décrit sur le plan de ce papier, avec un rayon $=a$, l'équation $y\sqrt{-1}=\mp\sqrt{xx-aa}$

exprime l'ordonnée d'une hyperbole équilatère décrite sur un plan perpendiculaire à celui de ce papier, et ayant a pour la valeur de chacun de ses deux demi-axes.

105. Soit ADBE (Fig. 12) le cercle dont l'ordonnée est $y = \pm\sqrt{aa - xx}$, et $AC = a$, le rayon. $AmnBpq$ est l'hyperbole équilatère dont l'ordonnée est $y = \pm\sqrt{xx - aa}\cdot\sqrt{-1}$, ou $y\sqrt{-1} = \mp\sqrt{xx - aa}$. Cette hyperbole est supposée décrite sur un plan perpendiculaire à celui du cercle, $\overline{AC}$, $\overline{CD}$, sont égaux à ses deux demi-axes. aCa et bCb sont ses assymptotes. Ces assymptotes font avec l'axe $\overline{AB}$ le même angle que les lignes $\overline{aCa}$, $\overline{bCb}$, font avec ce même axe, c'est-à-dire, un angle de 45°.

Or x et y coordonnées du cercle sont le sinus et le cosinus de l'arc qui leur correspond. Donc x et y coordonnées de l'hyperbole équilatère décrite sur un plan perpendiculaire à celui du cercle sont ce que deviennent le sinus et le cosinus, lorsque le sinus devient lui-même plus grand que le rayon.

106. Hyperbole équilatère.

Faisons tourner, par la pensée, la Figure 12 de manière que l'hyperbole $AmnBpq$ se trouve sur le plan de ce papier. Ce mouvement rendra le cercle ADBE perpendiculaire à ce même plan. Par là, l'ordonnée $y\sqrt{-1} = \mp\sqrt{xx - aa}$ de cette hyperbole cessant d'y être perpendiculaire, perd son signe $\sqrt{-1}$, et l'on a $y = \mp\sqrt{xx - aa}$, ou $yy = xx - aa$. Lorsque $x < a$, on a $yy = -(aa - xx)$, qui donne $y\sqrt{-1} = \pm\sqrt{aa - xx}$, équation à un cercle perpendiculaire au plan de ce papier.

L'équation $yy = -aa + xx$ donne $(a\sqrt{-1})^2 - (x\sqrt{-1})^2$,

qui diffère de l'équation (7) $y' = \left|\frac{a}{z}\sqrt{-1}\right|^2 - z^2$ (No. 43, Probl. II.) en ce que, dans cette dernière équation, l'abscisse z est réelle, au lieu que, dans l'équation $y' = (a\sqrt{-1})^2 - (x\sqrt{-1})^2$, l'abscisse $x\sqrt{-1}$ est imaginaire. Cela vient de ce que l'équation du No. 43 est celle d'un cercle décrit d'un mouvement conique, par une ligne brisée, au lieu que l'équation du présent numéro est celle d'un cercle décrit sur un plan perpendiculaire à celui de ce papier.

Ce numéro-ci et les deux précédens montrent que le cercle devient une hyperbole équilatère perpendiculaire, et l'hyperbole équilatère, un cercle perpendiculaire, lorsque leurs ordonnées deviennent imaginaires.

107. Ellipse et hyperbole non-équilatère.

Supposons que toute la Figure 12 tourne également autour de l'axe $\overline{AB}$, sans que ce papier tourne en même temps. Alors, au lieu d'un cercle et d'une hyperbole équilatère, on aura, d'abord par la projection du cercle sur ce papier, une ellipse, ensuite par la projection de l'hyperbole équilatère sur un plan perpendiculaire à ce papier, une hyperbole dont le grand axe sera $= \overline{AB}$, et dont le petit axe sera $< \overline{DE}$.

Les diamètres $\overline{ad}$, $\overline{bc}$, deviendront des diamètres conjugués égaux l'un à l'autre. Leur angle restera égal à l'angle asymptotique, quoiqu'alors $< 90°$.

108. L'équation de cette ellipse rapportée à ses *diamètres conjugués égaux* sera la même que celle du cercle, savoir, $yy = aa - xx$; mais ses coordonnées feront entr'elles un angle égal à celui des diamètres conjugués.

Il en sera de même des coordonnées de l'hyperbole rapportée à ses assymptotes.

On peut donc raisonner sur cette ellipse et cette hyperbole, comme nous l'avons fait sur le cercle et l'hyperbole équilatère. On y trouvera les mêmes analogies.

109. Parabole.

Son équation est $yy = px$. Si x est négatif, on a $-yy = px$, qui donne $y\sqrt{-1} = \pm px$. Cette seconde équation représente une seconde parabole qui est perpendiculaire au plan de ce papier et qui s'étend dans la partie négative de l'axe de la première parabole supposée décrite sur le plan de ce papier. Les sommets des deux paraboles se touchent, et les directions de leurs axes sont opposées.

110. Prenons maintenant une vue générale de toutes ces sections coniques à ordonnées imaginaires, que j'appellerai *appendices* des sections coniques.

1°. L'hyperbole équilatère est l'appendice du cercle. L'hyperbole non-équilatère est celui de l'ellipse, et la parabole celui de la parabole. Or comme les courbes-appendices sont perpendiculaires à leurs courbes originales, celles-ci le sont à celles-là. Donc le cercle est l'appendice de l'hyperbole équilatère, &c.

111. 2°. L'axe commun à la courbe originale et à la courbe-appendice est la projection de chacune de ces deux courbes sur le plan de son appendice.

112. 3°. Comme toutes ces courbes ont deux axes, savoir, celui des abscisses et celui des ordonnées, il faut dire relativement au second axe tout ce qui vient d'être dit relativement au premier.

113. 4°. Comme les courbes appendices ont les mêmes axes que leurs courbes originales, ces axes ont dans toute leur étendue, c'est-à-dire depuis $-\infty$ jusqu'à $+\infty$, des

ordonnées qui leur correspondent. Or il est facile de voir que ce que j'ai dit des sections coniques peut se dire également de toute autre courbe, soit algébrique, soit transcendante. Donc, supposant l'axe décrit sur le plan de ce papier et appellant *ordonnées* deux espèces de perpendiculaires à cet axe, savoir, celles qui sont sur ce papier et celles qui y sont perpendiculaires, je suis autorisé à établir cette proposition générale: *Une courbe quelconque a dans toute l'étendue de* CHACUN DE SES AXES, *c'est-à-dire, depuis* $-\infty$ *jusqu'à* $+\infty$, *autant d'ordonnées que le dégré de son équation contient d'unités, et comme le dégré d'une courbe transcendante est infini, le nombre de ces ordonnées dans tout le cours de chaque axe est infini.*

P. S. Depuis la composition de ce Mémoire, j'ai lû, dans le 1er. Tome des Mémoires de la Societé de Turin, un Mémoire de Mr. DE FONCENEX intitulé: *Réflexions sur les Quantités imaginaires*, où se trouve l'article suivant:

"No. 6. Si l'on réfléchit sur la nature des racines imaginaires qui, comme on sait, impliquent contradiction entre les données, on concevra évidemment qu'elles ne doivent point avoir de construction géométrique possible, puisqu'il n'est point de manière de les considérer qui lève la contradiction qui se trouve entre les données immuables par elles-même.

"Cependant, pour conserver une certaine analogie avec les quantités négatives, un auteur dont nous avons un cours d'algèbre d'ailleurs fort estimable a prétendu les devoir prendre sur une ligne perpendiculaire à celle où l'on les avoit supposées. Si, par exemple, on devoit couper la ligne

" CB (Fig. 1) $= 2a$ de façon que le rectangle des parties " $x \times (2a - x)$ fût égal à la quantité $2a^2$, on trouveroit " $x = a \pm \sqrt{-a^2}$. Pour trouver donc cette valeur de x, " qu'on prenne sur la ligne CB, la partie $AB = a$, partie " réelle de la valeur de x, et sur la perpendiculaire DD', les " parties AD, AD', aussi $= a$, on aura les points D, D', qui " resolvent le problème en ce que $BD \times DC$, ou $BD' \times DC' = 2a^2$; " mais puisque les points D, D', sont pris hors de la ligne CB, " et qu'une infinité d'autres points pris de même, auroient " aussi une propriété semblable, il est visible que, si cette " construction ne nous induit pas en erreur, elle ne nous fait " absolument rien connoître. C'est cependant là un des cas " où elle pourroit paroître plus spécieuse; car le plus souvent " on ne voit absolument pas comment le point trouvé pourroit " résoudre la question, quelque changemens qu'on se permît " dans l'énoncé du problème.

" Les racines imaginaires n'admettent donc pas une con- " struction géométrique, et on ne peut en tirer aucun avan- " tage dans la résolution des problèmes. On devroit par " conséquent s'attacher à les écarter autant qu'il est possible " des équations finales, puisque prises dans quelque sens que " ce soit, elles ne peuvent pas résoudre la question, comme " les racines négatives dont toute la contradiction consiste " dans leur manière d'être à l'égard des positives."

Voici les reflexions que cet article m'a suggérées.

1°. La question exposée par Mr. de Foncenex est à peu près la même que celle proposée par Mr. Carnot et qui est le 2d problème du présent écrit. Je ne connoissois pas les objections de Mr. de Foncenex; mais quand je les aurois

connues, j'aurois dit exactement tout ce que j'ai dit, et rien de plus.*

2°. Ecarter les quantités imaginaires des équations finales, comme Mr. de Foncenex le désire, est une chose impossible, passé le second dégré, à moins qu'on n'employe les sinus ou les cosinus, et je prouverai ailleurs que l'expression *finie* des sinus et des cosinus renferme, non seulement de fait, mais *nécessairement*, le signe $\sqrt{-1}$.†

On a vu aux Nos. 93 et suiv. comment la supposition d'un rayon portant le signe $\sqrt{-1}$ m'a conduit aux formules connues pour les équations du 3e. dégré, dont la première ne renferme plus ce signe. Si ce rayon ne portoit pas le signe $\sqrt{-1}$, alors ce signe reparoîtroit *nécessairement* dans cette première formule qui alors représenteroit le cas appellé *irréductible*.

3°. Dans les problèmes que je me suis proposé, je crois avoir fait voir que les racines imaginaires en donnoient des solutions qui n'étoient pas absurdes. On a même vû un problème du 3e. dégré, et par conséquent possible, qui ne pouvoit être résolu par la racine réelle, et qui l'étoit sans absurdité par les deux racines imaginaires.

4°. On a vu un problème (c'est le 3e.) qui paroissoit ab-

* Je n'ai pas cru devoir parler d'un Mémoire de Mr. Knox qui se trouve dans le 3e. tome des nouveaux Mémoires de Petersbourg, où ce Professeur entreprend de construire les quantités imaginaires, parcequ'il y suppose tacitement $\sqrt{-1} = -\sqrt{1}$.

† Mr. Woodhouse, dans un excellent Mémoire sur les quantités imaginaires inséré dans les *Transactions Philosophiques* de 1801, veut qu'on substitue aux signes *géométriques sin. cos. &c.* les expressions finies imaginaires dont je parle ici, et il regarde ces expressions qui renferment le signe $\sqrt{-1}$ comme représentatives de séries infinies *arithmétiques*. Tout cela est parfaitement conséquent au principe, que *l'algèbre n'est qu'une arithmétique universelle*.

surde et dont la solution (par des racines imaginaires) avoit un sens qui ne l'étoit pas. Ce problème n'étoit absurde qu'en apparence. C'étoit une énigme. Les racines imaginaires interprétées selon les principes de ce mémoire en ont donné le mot. Résoudre ainsi un problème, ce n'est pas en changer l'énoncé, c'est l'expliquer; ce n'est pas seulement répondre à une question proposée, c'est encore dire comment elle devoit l'être. Enfin, quelque sens qu'on ait en vue, en proposant une question qui mène à des racines imaginaires, ces racines y satisfont. Si c'est un sens raisonnable, elles le donnent et l'expliquent. Si c'est un sens absurde (ce qui n'arrive que quand on prend ces racines arithmétiquement) elles en montrent l'absurdité, mais elles ne l'expliquent pas; car ce qui est absurde est inexplicable.

5°. J'ai dit que, pour résoudre algébriquement une question, trois choses étoient nécessaires; 1°. traduire la question en langage algébrique; 2°. traduire l'énoncé algébrique de la question en sa solution algébrique; 3°. traduire cette solution algébrique en une solution éxécutive.

Rien ne fait plus d'honneur à l'esprit humain que la sagacité et l'addresse qu'on a mises dans la seconde espèce de ces traductions.

La méthode des variations de Mr. La Grange et l'usage qu'il en fait dans sa *mécanique analytique* offrent des modèles admirables de la première espèce; mais il y manque un point essentiel qu'il étoit impossible à l'algèbre-arithmétique d'obtenir. Je m'explique. Les questions de mécanique roulent sur des quantités concrètes. La manière même dont ces quantités sont concrètes fait partie de ces questions. Ainsi traduire ces quantités concrètes en quantités abstraites, c'est

les traduire imparfaitement. Delà l'impossibilité presque continuelle de parvenir, à l'aide de ces traductions, aux solutions demandées.* Dans les cas même où l'on a réussi, ce n'a été qu'avec une difficulté extrême. Quelle profondeur, quelle addresse, quelle sagacité n'a t'il pas fallu à Mr. La Place pour poser, avec le seul secours des équations différentielles connues (expliquées à la manière ordinaire) la dernière pierre à l'édifice Newtonien! Avec les seuls signes + et −, on ne peut traiter que la géométrie *mensurative*; cependant les problèmes de mécanique n'appartiennent pas plus à cette géométrie qu'à la géométrie *descriptive*.

A l'égard de la 3e. espèce de traduction qui n'est pas la moins importante, on s'en est beaucoup moins occupé que des deux autres. Aussi je puis citer un problème célèbre en mécanique (celui des *cordes vibrantes*) dont la solution, quoiqu'elle fasse le plus grand honneur aux géomètres célèbres qui l'ont donnée, n'est certainement pas complette. En effet, ils ont résolu l'équation aux différences partielles $\left(\frac{ddy}{dx^2}\right)=\left(\frac{ddy}{dt^2}\right)$, et leur solution donne la courbe indéterminée qui peut être formée par la longueur de la corde; mais ils ne résolvent pas l'équation $-\left(\frac{ddy}{dx^2}\right)=-\left(\frac{ddy}{dt^2}\right)$ qui est la véritable équation du problème et qui donneroit la courbe décrite par un point quelconque de cette corde, dans un plan perpendiculaire à son axe. L'équation $-\left(\frac{ddy}{dx^2}\right)=-\left(\frac{ddy}{dt^2}\right)$ est la même (arithmé-

* On a regardé comme un trait de génie dans Mr. La Grange d'être parvenu à traiter de la mécanique la plus sublime, sans figures. C'en a été un plus grand d'avoir vu qu'il n'en falloit pas. En effet, traiter la mécanique algébriquement, c'étoit (dans le système d'algèbre généralement adopté) ne la traiter qu'arithmétiquement. Or l'arithmétique n'a pas besoin des figures de la géométrie.

tiquement que l'équation $+\left(\frac{ddy}{dx^2}\right)=+\left(\frac{ddy}{dt^2}\right)$; mais elle n'est pas la même relativement à la géométrie descriptive. L'intégrale de la première est très différente de l'intégrale de la seconde. Elle est beaucoup plus bornée, (N°. 24,) à cause du signe — qui oblige de lui donner la forme $\left\{\frac{ddy}{(dx\sqrt{-1})(dx\sqrt{-1})}\right\}=\left\{\frac{ddy}{(dt\sqrt{-1})(dt\sqrt{-1})}\right\}$. La solution qu'on tire de cette dernière équation est à peu près semblable à celle que Newton a donnée d'un problème analogue, dans la 47e. proposition du second livre de ses Principes, et qui a paru peu satisfaisante aux géomètres qui sont venus depuis ce grand homme. Oserois-je ajouter qu'ils n'en ont pas saisi le vrai sens ?

L'explication que j'ai donnée du signe $\sqrt{-1}$ me paroît devoir applanir bien des difficultés.

Fig. 1.

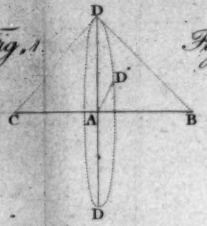


Fig. 2.

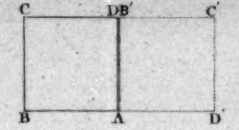


Fig. 3.

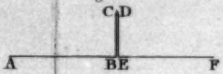


Fig. 11.

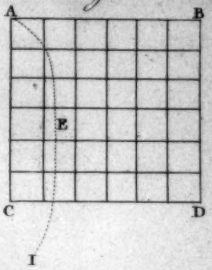


Fig. 4.

Fig. 5.

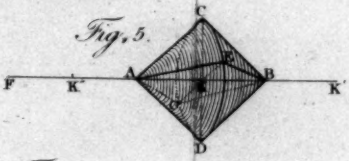


Fig. 7.

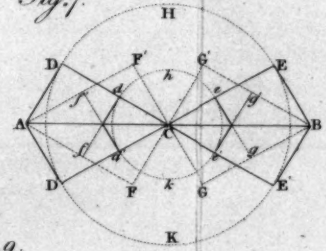


Fig. 6.

Fig. 12.

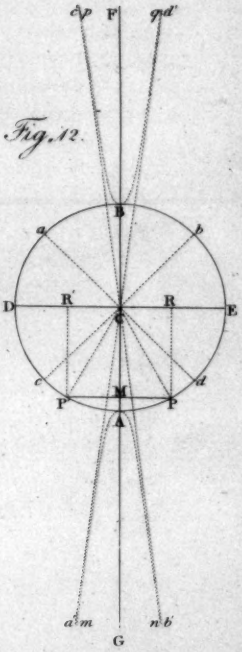


Fig. 9.

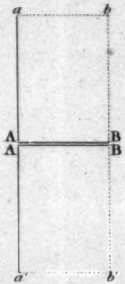


Fig. 8.

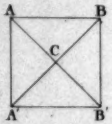
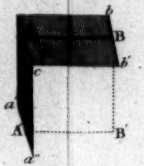


Fig. 10.



IV. *Chemical Experiments on Guaiacum.* By Mr. William Brande. *Communicated by* Charles Hatchett, *Esq. F.R.S.*

Read December 19, 1805.

Among the numerous substances which are comprehended under the name of resins, there is perhaps no one which possesses so many curious properties, as that now under consideration; and it is remarkable that no more attention has been paid to the subject, since many of the alterations which it undergoes when treated with different solvents, have been mentioned by various authors.

§ I.

Guaiacum has a green hue externally; is in some degree transparent; and breaks with a vitreous fracture.

When pulverised it is of a gray colour, but gradually becomes greenish on exposure to air.

It melts when heated, and diffuses at the same time a pungent aromatic odour.

It has when in powder a pleasant balsamic smell, but scarcely any taste, although when swallowed it excites a very powerful burning sensation in the throat.

Its specific gravity is 1.2289.

§ II.

1. When pulverised guaiacum is digested in a moderate heat with distilled water, an opaque solution is formed, which becomes clear on passing the whole through a filter.

The filtrated liquor is of a greenish-brown colour; it has a peculiar smell, and a sweetish taste.

It leaves on evaporation a brown substance, which is soluble in alcohol, nearly soluble in boiling water, and very little acted upon by sulphuric ether.

This solution was examined by the following re-agents.

Muriate of alumina occasioned a brown insoluble precipitate after some hours had elapsed.

Muriate of tin formed a brown flaky precipitate under the same circumstances.

Nitrate of silver gave a copious brown precipitate.

Suspecting the presence of lime in the solution, I added a few drops of oxalate of ammonia, when the liquid immediately became turbid, and deposited brown flakes, which, after having been treated with boiling alcohol, yielded traces of oxalate of lime.

These effects, therefore, indicate the presence of a substance in guaiacum, which possesses the properties of extract;* the action of the reagent is however somewhat modified, by a small quantity of lime which is also in solution.

One hundred grains of guaiacum yielded about nine grains of this impure extractive matter.

* By the term extract, I mean that substance, which by chemists is called the Extractive Principle of Vegetables. Vide THOMSON's Syst. of Chemistry, 2d edit. Vol. IV. p. 276.

1. Alcohol dissolves guaiacum with facility, leaving some extraneous matter, which generally amounts to about 5 *per cent.*

This solution is of a deep brown colour; the addition of water separates the resin, forming a milky fluid which passes the filter.

Acids produce the following changes:

A. Muriatic acid throws down an ash-coloured precipitate, which is not re-dissolved by heating the mixture. In this case the resin appears but little altered.

B. Liquid oxy-muriatic acid when poured into this solution, forms a precipitate of a very beautiful pale-blue colour, which may be preserved unaltered.

C. Sulphuric acid, when not added in too large a quantity, separates the resin of a pale green colour.

D. Acetic acid does not form any precipitate. This acid is indeed capable of dissolving most of the resins.

E. Nitric acid diluted with one-fourth of its weight of water, causes no precipitate till after the period of some hours. The liquid at first assumes a green colour, and if water be added at this period, a green precipitate may be obtained; the green colour soon changes to blue, (when by the same means a blue precipitate may be obtained;) it then becomes brown, and a brown precipitate spontaneously makes its appearance, the properties of which will be afterwards mentioned.

The changes of colour produced by nitric, and oxy-muriatic acids, in the alcoholic solution, are very remarkable, and I believe peculiar to guaiacum: there is moreover much reason to suppose that the above alterations in colour are occasioned

by oxygen.* It likewise appears from that which has been stated, that the blue and green oxides (if they may be so called by way of distinction) are soluble in the mixture of nitric acid and alcohol, while the brown precipitate is insoluble.

F. Alkalis do not form any precipitate when added to the solution of guaiacum in alcohol.

8. Guaiacum is less soluble in sulphuric ether than in alcohol; the properties of this solution nearly coincide with those just mentioned.

4. Muriatic acid dissolves a small portion of guaiacum, the solution assuming a deep brown colour; but if heat be applied, the resin melts into a blackish mass, preventing any farther action from taking place.

5. Sulphuric acid forms with guaiacum a deep red liquid, which, when fresh prepared, deposits a lilac coloured precipitate on the addition of water; a precipitate is also formed

* The following experiments appear to verify this supposition:

Fifty grains of freshly pulverised guaiacum were introduced into a glass jar containing 60 cubic inches of oxy-muriatic acid gas. The resin speedily assumed a brown colour, having passed through several shades of green and blue. Liquid ammonia was poured on this brown substance, while yet immersed in the acid; the whole became green; it therefore seemed thus to be deprived of part of the oxygen which it apparently had acquired by the preceding experiment. An equal portion of the same guaiacum was exposed under similar circumstances to the action of oxy-muriatic acid, excepting that the glass in which the experiment was made, was covered with a black varnish, and placed in a dark apartment. On examining the result of this experiment, the resin was found to have undergone precisely the same changes as when exposed to light. Ammonia had also the same effect.

Guaiacum was also exposed over mercury to oxygen gas; the resin assumed after some days the green colour which a longer exposure to the atmosphere produces: this change was likewise found by a second experiment to be effected without the presence of light.

by the alkalis. If heat be employed in forming this solution, the resin is speedily decomposed; and if the whole of the acid be evaporated, there remains a black coaly substance, together with some sulphate of lime.

6. Nitric acid appears to exert a more powerful action on guaiacum than on any of the resinous bodies.

100 grains of pure guaiacum previously reduced to powder, were cautiously added to two ounces of nitric acid, of the specific gravity of 1.39. The resin at first assumed a dark green colour, a violent effervescence was produced, attended with the emission of much nitrous gas, and the whole was dissolved without the assistance of heat, which is not the case with the resins in general, for when these bodies are thus treated with nitric acid, they are commonly converted into an orange-coloured porous mass.

The solution thus formed, yielded while recent, a brown precipitate with the alkalis, which was redissolved on the application of heat, forming a deep brown liquid.

Muriatic acid also separated the guaiacum from this solution, not however without having undergone some change.

Sulphuric acid caused no precipitate.

After this solution of guaiacum in nitric acid had remained undisturbed for some hours, a considerable proportion of crystallised oxalic acid was deposited.

When guaiacum was treated with dilute nitric acid, the results were somewhat different. A slight effervescence took place, and part of the resin was dissolved, the remainder being converted into a brown substance, resembling the precipitate obtained from the alcoholic solution as above mentioned. (2. E.)

This brown substance appears to be guaiacum, the properties of which are materially altered, by its combination with oxygen; and I am led to think that the changes of colour produced by nitric and oxy-muriatic acids, are the consequence of the different proportions of oxygen with which the guaiacum has been united; for we know that the colours of metallic, and many other bodies, are greatly influenced by the same cause.

The brown substance was separated by filtration; the filtrated liquor yielded yellow flocculent precipitates with the alkalis, and on examination was found to hold nitrate of lime in solution.

The undissolved portion was of a deep chocolate-brown colour. A similar substance may also be obtained, by evaporating the recent nitric solution to dryness, taking care not to apply too much heat towards the end of the process.

The substance obtained by either of these means, possesses the properties of a resin in greater perfection than guaiacum; it is equally soluble in alcohol and sulphuric ether, insoluble in water, &c.; but when burned it emits a peculiar smell, more resembling animal than vegetable bodies. If, however, fresh portions of nitric acid be added three or four times successively; or if a large quantity be employed to form the solution; the product obtained by evaporation is then of a very different nature; for it has lost all the characteristic properties of a resin, having become equally soluble in water and alcohol; the solution of it in this state having an astringent bitter taste.*

* Vide Mr. HATCHETT's two Papers on an artificial Substance which possesses the principal characteristic Properties of Tannin. Phil. Trans. 1803, p. 211, and 283.

7. Guaiacum is copiously soluble in the pure and carbonated alkalis, forming greenish-brown liquids.

Two ounces of a saturated solution of caustic potash took up rather more than 65 grains of the resin; the same quantity of liquid ammonia dissolved only 25 grains.

Nitric acid formed in these solutions a deep brown precipitate, the shades of which varied according to the quantity of acid which had been employed.

This precipitate was found on examination to possess the properties of that formed by nitric acid in the solution of guaiacum (2. E.) in alcohol.

Dilute sulphuric acid, when poured into any of the above alkaline solutions, formed a flesh-coloured curdy precipitate. Muriatic acid produced the same effect.

The two last mentioned precipitates differ from guaiacum, in being less acted upon by sulphuric ether and more soluble in boiling water, their properties therefore approach nearer to extract. Moreover, when these precipitates were redissolved in ammonia, and were again separated by muriatic acid, the above mentioned properties became more evident.

§ III.

100 grains of very pure guaiacum in powder, were put into a glass retort, to which the usual apparatus was adapted. The distillation was gradually performed on an open fire, until the bottom of the retort became red hot.

The following products were obtained:

	Grains.
Acidulated water - - - -	5.5
Thick brown oil, becoming turbid on cooling	24.5
Thin empyreumatic oil - - -	30.0
Coal remaining in the retort - - -	30.5
Mixed gases, consisting chiefly of carbonic acid and carbonated hydrogen - - -	9.0
	99.5

The coal, amounting to 30.5 grains, yielded on incineration 3 grains of lime. To discover whether any fixed alkali was present, 200 grains of the purest guaiacum (that in drops) were reduced to ashes; these were dissolved in muriatic acid, and precipitated by ammonia; the whole was then filtrated, and the clear liquor evaporated to dryness, but not any trace of a neutral salt with a basis of fixed alkali was perceptible.

§ IV.

From the action of different solvents on guaiacum, it appears, that although this substance possesses many properties in common with resinous bodies, it nevertheless differs from them in the following particulars:

1. By affording a portion of vegetable extract.
2. By the curious alterations which it undergoes when

subjected to the action of bodies, which readily communicate oxygen, such as nitric and oxy-muriatic acids; and the rapidity with which it dissolves in the former.

3. By being converted into a more perfect resin; in which respect guaiacum bears some resemblance to the green resin which constitutes the colouring matter of the leaves of trees, &c.*

4. By yielding oxalic acid.

5. By the quantity of charcoal and lime which are obtained from it when subjected to destructive distillation.

§ V.

From the whole therefore of the above mentioned properties, it evidently appears that guaiacum is a substance very different from those which are denominated resins, and that it is also different from all those which are enumerated amongst the balsams, gum resins, gums, and extracts: most probably it is a substance distinct in its nature from any of the above, in consequence of certain peculiarities in the proportions and chemical combination of its constituent elementary principles; but as this opinion may be thought not sufficiently supported by the facts which have been adduced, we may for the present be allowed to regard guaiacum as composed of a resin modified by the vegetable extractive

* This substance was found by PROUST to be insoluble in water, and soluble in alcohol. When treated with oxy-muriatic acid, it assumed the colour of a withered leaf, acquiring the resinous properties in greater perfection. Vide THOMSON's Syst. of Chemistry, 2d edit. Vol. IV. p. 318.

principle, and as such, perhaps the definition of it by the term of an *Extracto-Resin* may be adopted without impropriety.

P. S. I have observed that the action of oxygen on some of the other resinous bodies is very remarkable. It is well known that by digesting mastich in alcohol, a partial solution only is formed, and there remains an elastic substance, which is generally said to possess the properties of pure caoutchouc; it appears however to differ from this substance in becoming hard when dried by exposure to air. Moreover, I have remarked that the part of mastich which remains dissolved by alcohol, may be again precipitated by water, and, on examination, I found the precipitate to possess the properties of a pure resin: but when a stream of oxy-muriatic acid gas was made to pass through the solution, a tough elastic substance was thrown down, which became brittle when dried, and was soluble in boiling alcohol, but separated again as the solution cooled: its properties, therefore, somewhat approached to those of the original insoluble part.

V. *On the Direction of the Radicle and Germen during the Vegetation of Seeds.* By Thomas Andrew Knight, *Esq. F. R. S. In a Letter to the Right Hon. Sir* Joseph Banks, *K. B. P. R. S.*

Read January 9, 1806.

MY DEAR SIR,

It can scarcely have escaped the notice of the most inattentive observer of vegetation, that in whatever position a seed is placed to germinate, its radicle invariably makes an effort to descend towards the centre of the earth, whilst the elongated germen takes a precisely opposite direction; and it has been proved by Du Hamel* that if a seed, during its germination, be frequently inverted, the points both of the radicle and germen will return to the first direction. Some naturalists have supposed these opposite effects to be produced by gravitation; and it is not difficult to conceive that the same agent, by operating on bodies so differently organized as the radicle and germen of plants are, may occasion the one to descend and the other to ascend.

The hypothesis of these naturalists does not, however, appear to have been much strengthed by any facts they were able to adduce in support of it, nor much weakened by the arguments of their opponents; and therefore, as the phenomena

* *Physique des Arbres.*

observable during the conversion of a seed into a plant are amongst the most interesting that occur in vegetation, I commenced the experiments, an account of which I have now the honour to request you to lay before the Royal Society.

I conceived that if gravitation were the cause of the descent of the radicle, and of the ascent of the germen, it must act either by its immediate influence on the vegetable fibres and vessels during their formation, or on the motion and consequent distribution of the true sap afforded by the cotyledons: and as gravitation could produce these effects only whilst the seed remained at rest, and in the same position relative to the attraction of the earth, I imagined that its operation would become suspended by constant and rapid change of the position of the germinating seed, and that it might be counteracted by the agency of centrifugal force.

Having a strong rill of water passing through my garden, I constructed a small wheel similar to those used for grinding corn, adapting another wheel of a different construction, and formed of very slender pieces of wood, to the same axis. Round the circumference of the latter, which was eleven inches in diameter, numerous seeds of the garden bean, which had been soaked in water to produce their greatest degree of expansion, were bound, at short distances from each other. The radicles of these seeds were made to point in every direction, some towards the centre of the wheel, and others in the opposite direction; others as tangents to its curve, some pointing backwards, and others forwards, relative to its motion; and others pointing in opposite directions in lines parallel with the axis of the wheels. The whole was inclosed in a box, and secured by a lock, and a wire grate was placed

to prevent the ingress of any body capable of impeding the motion of the wheels.

The water being then admitted, the wheels performed something more than 150 revolutions in a minute; and the position of the seeds relative to the earth was of course as often perfectly inverted, within the same period of time; by which I conceive that the influence of gravitation must have been wholly suspended.

In a few days the seeds began to germinate, and as the truth of some of the opinions I had communicated to you, and of many others which I had long entertained, depended on the result of the experiment, I watched its progress with some anxiety, though not with much apprehension; and I had soon the pleasure to see that the radicles, in whatever direction they were protruded from the position of the seed, turned their points outwards from the circumference of the wheel, and in their subsequent growth receded nearly at right angles from its axis. The germens, on the contrary, took the opposite direction, and in a few days their points all met in the centre of the wheel. Three of these plants were suffered to remain on the wheel, and were secured to its spokes to prevent their being shaken off by its motion. The stems of these plants soon extended beyond the centre of the wheel: but the same cause, which first occasioned them to approach its axis, still operating, their points returned and met again at its centre.

The motion of the wheel being in this experiment vertical, the radicle and germen of every seed occupied, during a minute portion of time in each revolution, precisely the same position they would have assumed had the seeds vegetated at

rest; and as gravitation and centrifugal force also acted in lines parallel with the vertical motion and surface of the wheel, I conceived that some slight objections might be urged against the conclusions I felt inclined to draw. I therefore added to the machinery I have described another wheel, which moved horizontally over the vertical wheels; and to this, by means of multiplying wheels of different powers, I was enabled to give many different degrees of velocity. Round the circumference of the horizontal wheel, whose diameter was also eleven inches, seeds of the bean were bound as in the experiment, which I have already described, and it was then made to perform 250 revolutions in a minute. By the rapid motion of the water-wheel much water was thrown upwards on the horizontal wheel, part of which supplied the seeds upon it with moisture, and the remainder was dispersed, in a light and constant shower, over the seeds in the vertical wheel, and on others placed to vegetate at rest in different parts of the box.

Every seed on the horizontal wheel, though moving with great rapidity, necessarily retained the same position relative to the attraction of the earth; and therefore the operation of gravitation could not be suspended, though it might be counteracted, in a very considerable degree, by centrifugal force: and the difference, I had anticipated, between the effects of rapid vertical and horizontal motion soon became sufficiently obvious, The radicles pointed downwards about ten degrees below, and the germens as many degrees above, the horizontal line of the wheel's motion; centrifugal force having made both to deviate 80 degrees from the perpendicular direction each would have taken, had it vegetated at rest. Gradually

diminishing the rapidity of the motion of the horizontal wheel, the radicles descended more perpendicularly, and the germens grew more upright; and when it did not perform more than 80 revolutions in a minute, the radicle pointed about 45 degrees below, and the germen as much above, the horizontal line, the one always receding from, and the other approaching to, the axis of the wheel.

I would not, however, be understood to assert that the velocity of 250, or of 80 horizontal revolutions in a minute will always give accurately the degrees of depression and elevation of the radicle and germen which I have mentioned; for the rapidity of the motion of my wheels was sometimes diminished by the collection of fibres of conferva against the wire grate; which obstructed in some degree the passage of the water: and the machinery, having been the workmanship of myself and my gardener, can not be supposed to have moved with all the regularity it might have done, had it been made by a professional mechanic. But I conceive myself to have fully proved that the radicles of germinating seeds are made to descend, and their germens to ascend, by some external cause, and not by any power inherent in vegetable life: and I see little reason to doubt that gravitation is the principal, if not the only agent employed, in this case, by nature. I shall therefore endeavour to point out the means by which I conceive the same agent may produce effects so diametrically opposite to each other.

The radicle of a germinating seed (as many naturalists have observed) is increased in length only by new parts successively added to its apex or point, and not at all by any

general extension of parts already formed: and the new matter which is thus successively added unquestionably descends in a fluid state from the cotyledons.* On this fluid, and on the vegetable fibres and vessels whilst soft and flexible, and whilst the matter which composes them is changing from a fluid to a solid state, gravitation, I conceive, would operate sufficiently to give an inclination downwards to the point of the radicle; and as the radicle has been proved to be obedient to centrifugal force, it can scarcely be contended that its direction would remain uninfluenced by gravitation.

I have stated that the radicle is increased in length only by parts successively added to its point: the germen, on the contrary, elongates by a general extension of its parts previously organized; and its vessels and fibres appear to extend themselves in proportion to the quantity of nutriment they receive. If the motion and consequent distribution of the true sap be influenced by gravitation, it follows, that when the germen at its first emission, or subsequently, deviates from a perpendicular direction, the sap must accumulate on its under side: and I have found in a great variety of experiments on the seeds of the horse chesnut, the bean, and other plants, when vegetating at rest, that the vessels and fibres on the under side of the germen invariably elongate much more rapidly than those on its upper side; and thence it follows that the point of the germen must always turn upwards. And it has been proved that a similar increase of growth takes place on the external side of the germen when the sap

* See Phil. Trans. of 1805.

is impelled there by centrifugal force, as it is attracted by gravitation to its under side, when the seed germinates at rest.

This increased elongation of the fibres and vessels of the under side is not confined to the germens, nor even to the annual shoots of trees, but occurs and produces the most extensive effects in the subsequent growth of their trunks and branches. The immediate effect of gravitation is certainly to occasion the further depression of every branch, which extends horizontally from the trunk of the tree; and, when a young tree inclines to either side, to increase that inclination: but it at the same time, attracts the sap to the under side, and thus occasions an increased longitudinal extension of the substance of the new wood on that side.* The depression of the lateral branch is thus prevented; and it is even enabled to raise itself above its natural level, when the branches above it are removed; and the young tree, by the same means, becomes more upright, in direct opposition to the immediate action of gravitation: nature, as usual, executing the most important operations by the most simple means.

I could adduce many more facts in support of the preceding deductions, but those I have stated, I conceive to be sufficiently conclusive. It has however been objected by Du Hamel, (and the greatest deference is always due to his opinions,) that gravitation could have little influence on the direction of the germen, were it in the first instance protruded, or were it subsequently inverted, and made to

* This effect does not appear to be produced in what are called weeping trees; the cause of which I have endeavoured to point out in a former Memoir. Phil. Trans. 1804.

point perpendicularly downwards. To enable myself to answer this objection, I made many experiments on seeds of the horse chesnut, and of the bean, in the box I have already described; and as the seeds there were suspended out of the earth, I could regularly watch the progress of every effort made by the radicle and germen to change their positions. The extremity of the radicle of the bean, when made to point perpendicularly upwards, generally formed a considerable curvature within three or four hours, when the weather was warm. The germen was more sluggish; but it rarely or never failed to change its direction in the course of twenty-four hours; and all my efforts to make it grow downwards, by slightly changing its direction, were invariably abortive.

Another, and apparently a more weighty, objection to the preceding hypothesis, (if applied to the subsequent growth and forms of trees,) arises from the facts that few of their branches rise perpendicularly upwards, and that their roots always spread horizontally; but this objection I think may be readily answered.

The luxuriant shoots of trees, which abound in sap, in whatever direction they are first protruded, almost uniformly turn upwards, and endeavour to acquire a perpendicular direction; and to this their points will immediately return, if they are bent downwards during any period of their growth; their curvature upwards being occasioned by an increased extension of the fibres and vessels of their under sides, as in the elongated germens of seeds. The more feeble and slender shoots of the same trees will, on the contrary, grow in almost every direction, probably because their fibres, being more dry, and their vessels less amply supplied with sap, they are

less affected by gravitation. Their points, however, generally shew an inclination to turn upwards; but the operation of light, in this case, has been proved by BONNET* to be very considerable.

The radicle tapers rapidly, as it descends into the earth, and its lower part is much compressed by the greater solidity of the mould into which it penetrates. The true sap also continues to descend from the cotyledons and leaves, and occasions a continued increase of the growth of the upper parts of the radicle, and this growth is subsequently augmented by the effects of motion, when the germen has risen above the ground. The true sap is therefore necessarily obstructed in its descent; numerous lateral roots are generated, into which a portion of the descending sap enters. The substance of these roots, like that of the slender horizontal branches, is much less succulent than that of the radicle first emitted, and they are in consequence less obedient to gravitation: and therefore meeting less resistance from the superficial soil, than from that beneath it, they extend horizontally in every direction, growing with most rapidity, and producing the greatest number of ramifications, wherever they find most warmth, and a soil best adapted to nourish the tree. As these horizontal, or lateral, roots surround the base of the tree on every side, the true sap descending down its bark, enters almost exclusively into them, and the first perpendicular root, having executed its office of securing moisture to the plant, whilst young, is thus deprived of proper nutriment, and, ceasing almost wholly to grow, becomes of no importance to the tree. The tap root of the oak, about which so

* *Récherches sur l'Usage des Feuilles dans les Plantes.*

much has been written, will possibly be adduced as an exception; but having attentively examined at least 20,000 trees of this species, many of which had grown in some of the deepest and most favourable soils of England, and never having found a single tree possessing a tap root, I must be allowed to doubt that one ever existed.

As trees possess the power to turn the upper surfaces of their leaves, and the points of their shoots to the light, and their tendrils in any direction to attach themselves to contiguous objects, it may be suspected that their lateral roots are by some means directed to any soil in their vicinity which is best calculated to nourish the plant, to which they belong; and it is well known that much the greater part of the roots of an aquatic plant, which has grown in a dry soil, on the margin of a lake or river, have been found to point to the water; whilst those of another species of tree which thrives best in a dry soil, have been ascertained to take an opposite direction: but the result of some experiments I have made is not favourable to this hypothesis, and I am rather inclined to believe that the roots disperse themselves in every direction, and only become most numerous where they find most employment, and a soil best adapted to the species of plant. My experiments have not, however, been sufficiently varied, or numerous, to decide this question, which I propose to make the subject of future investigation.

I am, &c.

T. A. KNIGHT.

Elton, Nov. 22, 1805.

VI. *A third Series of Experiments on an artificial Substance, which possesses the principal characteristic Properties of Tannin; with some Remarks on Coal.* By Charles Hatchett, *Esq. F. R. S.*

Read January 16, 1806.

§ I.

In my former Papers upon this subject, some account has been given of the effects produced by sulphuric acid upon turpentine, resin, and camphor; and I shall now state the results of other experiments made with the same acid upon a great number of the resins, balsams, gum resins, and gums, the greater part of which, afforded that modification of the artificial tanning substance, which for the sake of distinction, I have in the preceding papers denominated the third variety.

The process was simple digestion in sulphuric acid, after which, the residuum was well edulcorated, and was then digested in alcohol. This was separated by distillation, the dry substance which remained was infused in cold distilled water, and the portion dissolved, was examined by solution of isinglass, muriate of tin, acetite of lead, and sulphate of iron.

Much sulphureous acid, carbonic acid, several of the vegetable acids, particularly benzoic acid, (when the balsams were employed,) and apparently water, were produced during the operation; but in this Paper I shall only notice two of the products, namely, the tanning substance and the coal.

The sulphuric acid almost immediately dissolved the resins, and formed transparent brown solutions, which progressively became black.

The same effect was produced on most of the other substances; but the solutions of the balsams and of guaiacum were at first of a deep crimson, slightly inclining to brown.

Caoutchouc and elastic bitumen were not dissolved, but after having been digested for more than two months, were only superficially carbonized.

The gums and the saccharine substances required many evaporations and filtrations before the whole of their carbonaceous residua could be obtained.

These were the principal effects observed during the experiments, and I have stated them in this manner, that tedious and useless repetitions may be avoided.

§ II.

Turpentine, common resin, elemi, tacamahac, mastich, copaiba, copal, camphor, benzoin, balsam of Tolu, balsam of Peru, asa fœtida, and amber, yielded an abundance of the tanning substance.

Oil of turpentine also afforded much of it; asphaltum yielded a small portion; some slight traces of it were even obtained from gum arabic and tragacanth; but none was produced by guaiacum, dragon's blood, myrrh, gum ammoniac, olibanum, gamboge, caoutchouc, elastic bitumen, liquorice, and manna. I am persuaded, however, that many of these would have afforded the tanning substance had not the digestion been of too long a duration.

Olive oil was partly converted into the above mentioned

substance, and also linseed oil, wax, and animal fat; but the three last appear to merit some attention.

Linseed Oil.

This oil with sulphuric acid very soon formed a thick blackish-brown liquid, which after being long digested in a sand-bath, was still partly soluble in cold water, and passed the filter. This solution precipitated gelatine; the residuum was a tough black substance, which became hard on exposure to air. A great part was soluble in alcohol, and formed a brown liquid, which became turbid by the addition of water. When this was evaporated, a brown substance remained, which was partially dissolved by cold water, and the solution thus formed, was rendered turbid by gelatine.

The undissolved portion left by the alcohol, was of a blackish-brown; it was soft and tenacious, and appeared to retain many of the properties of an inspissated fat oil.

Bleached Wax.

That which was employed in this experiment, was the white wax of the shops, which is sold in the form of small round cakes. It formed with sulphuric acid a thick black magma, and was not acted upon by cold distilled water when washed with it upon a filter. Upon being digested with alcohol in a sand-bath, a brownish solution was formed, which upon cooling became very turbid, and appeared as if filled with a white flocculent substance. The same operation was repeated with different portions of alcohol until this ceased to act. The whole of the solutions in alcohol were

then mixed, a large quantity of distilled water was added, and the alcohol was separated by distillation.

On the surface of the remaining liquor, when cold, a white crust was formed, which being separated, was found to possess the properties of spermaceti, and weighed 18 grains. The filtrated liquor was then evaporated to a small quantity, became of a pale brown colour, and was rendered turbid by solution of isinglass.

Animal Fat.

This experiment was made upon the kidney fat of veal, but I cannot take upon me to assert that the results would have been the same with every kind of fat. 100 grains of it with one ounce of concentrated sulphuric acid, after some time, formed a blackish soft mass; a second ounce of sulphuric acid was then added, and the whole was digested and occasionally heated during nearly three months. Six ounces of distilled water were poured upon the black pulpy mass, and formed a thick uniform liquid, which, after digestion for six or seven days, was when cold filtrated. The liquor which passed was of a brown colour, and upon evaporation became black, leaving a considerable portion of a blackish substance upon the filter, which was added to that which had been collected by the first filtration. The whole was washed with cold water, which passed colourless. Boiling water was then poured upon the filter, by which a considerable portion was rapidly dissolved, and a brownish-black solution was formed, which copiously precipitated gelatine.

The residuum on the filter was then dried, and being col-

lected, was digested in alcohol, which dissolved the greater part.

The solution in alcohol was filtrated, but (apparently by the effect of air) a considerable deposit was formed on the filter, which was again dissolved by alcohol. Water rendered the solution turbid, and a black light flaky substance, which weighed 41 grains, remained upon the filter. The filtrated liquor was then evaporated, and left a grayish-black substance, which weighed 50 grains. This last substance was highly inflammable, and when burned, emitted a very peculiar odour, resembling partly that of fat and partly that of asphaltum. It easily melted, and also immediately dissolved in cold alcohol, from which, like the resinous substances, it was precipitated by water.

The black light flaky residuum, which weighed 41 grains, was found to consist partly of the substance above mentioned and partly of coal, but the proportion of this last was not ascertained.

Coagulated albumen and prepared muscular fibre were also separately exposed to the action of sulphuric acid in the manner above described, but did not afford any substance by which gelatine could be precipitated, coal being the only product which remained.

Almost every one of the bodies which have been employed in these experiments, seem to be in some measure different in respect to the progressive effects produced upon them by sulphuric acid; and all other circumstances being similar, there appears to be a certain period of the process when the production of the tanning substance has arrived at its maximum, after which, a gradual diminution of it takes place, and

at length total destruction. These effects are produced at different periods, according to the substance which may be the subject of the experiment, and therefore it is impossible at present to state the utmost quantity of the tanning substance which, under equal circumstances, may be obtained from each of the resins, balsams, &c.

The tanning substance appears to be always the same, whether obtained from turpentine, or common resin, or from the balsams, or from asa fœtida, or camphor, or indeed from any of the bodies which have been enumerated; its effects on the different reagents are similar: by the addition of a small portion of nitric acid, and subsequent evaporation, it is converted into that which I have called the first variety; or if digested with sulphuric acid, it is speedily destroyed, and becomes mere coal. In the latter case, therefore, the same agent which at first produced it becomes at length the cause of its destruction, and thus we find, that although a tanning substance may be obtained from resinous and other bodies by means of sulphuric and by nitric acid, yet in the former case the product is variable, and is formed at or about the mean period of the operation, whilst the latter is an ultimate and invariable effect, beyond which, no apparent change can be produced by any continuation of the process.*

§ III.

I have already stated, that caoutchouc, and elastic bitumen, were only superficially acted upon when digested for a very

* In the former Papers upon this subject I have observed, that the tanning substance produced by sulphuric acid, is very inferior in energy to that, which is formed by nitric acid.

long time in sulphuric acid; and it is remarkable, that these substances, which in their external characters so much resemble each other, should be similar in their habits when exposed to the effects of this acid; for, unlike the resins and most of the other bodies which were subjected to the preceding experiments, and which were almost immediately dissolved when the acid was poured upon them, these on the contrary remained undissolved, and only became partially carbonized on their surfaces. Even nitric acid does not so rapidly effect a change in the elastic bitumen as it does when applied to the other bituminous substances.

1.

100 grains of pure soft elastic bitumen were digested during three weeks in one ounce of nitric acid, diluted with an equal quantity of water; a tough and slightly elastic orange-coloured mass then remained. Another ounce of the acid, not diluted, was poured upon this mass, and the digestion was continued until the whole was evaporated. The residuum was tenacious, and of the colour above mentioned. Water partially dissolved it, and formed a deep yellow liquid, which copiously precipitated gelatine, and possessed the other properties of the tanning substance which is produced from the resins, &c. by nitric acid.

An orange-coloured mass still remained, which was speedily dissolved by alcohol, and was precipitated from it by a large addition of water.

This substance in many of its properties resembled the resins, but in others, seemed to approach those which characterize the vegetable extractive matter. It appeared to

be similar to that which has been cursorily mentioned in my first Paper, and which was obtained from many of the pit-coals and bitumens when treated with nitric acid. I have since paid more attention to this substance during the following experiments:

Kilkenny coal was digested with nitric acid, and progressively, although with difficulty, was converted into that variety of the tanning substance which has so often been mentioned. Similar experiments were made on the same sort of coal from Wales, which was given to me by my friend Mr. TENNANT, as well as upon a coal sent to me by Professor WOODHOUSE, which was from Pensylvania, and is there called Leigh high coal. All of these were converted into the tanning substance, but they did not yield any product similar to that obtained from the elastic bitumen.

The contrary however happened when the common pit-coal, or Cannel coal, or asphaltum, were employed. For when these were treated in the way which has been described, and when the digestion was not too long continued, then I obtained from 100 grains of each of the above substances (after the separation of the tanning matter) a residuum as follows:

From 100 grains of the common Newcastle coal 9 grains.
From 100 grains of Cannel coal - - - 56 grains.
From 100 grains of pure asphaltum - - 87 grains.

The substances thus obtained, were very similar in their external characters, being of a pale brown, approaching to Spanish snuff colour; their internal fracture was dark brown, with a considerable degree of resinous lustre. When exposed to heat they did not easily melt, but as soon as inflamed, they emitted a resinous odour mixed with that of fat oil, and pro-

duced a very light coal, much exceeding the bulk of the original substance.

Alcohol completely dissolved them, and if water in a large proportion was added to a saturated solution, a precipitate was obtained, but after each precipitation, a portion always remained dissolved by the water, which acted upon the different reagents in a manner similar to the solutions of vegetable extractive matter. The flavour was also bitter, and in some degree aromatic, so that the residua, whether obtained from pit-coal, from Cannel coal, or from asphaltum, seemed to possess properties intermediate between those of resin, and those of the vegetable extractive substance. They appeared however, to be removed only by a very few degrees from the tanning substance; for if digested in a small quantity of nitric acid, and subsequently evaporated, they were immediately converted into it; or if digested with sulphuric acid, they speedily became reduced to coal.

§ IV.

In the 5th Section of my second Paper, some remarks were made on the decoctions obtained from vegetable substances which had been previously roasted; and although (excepting one instance) these decoctions did not afford any permanent precipitate with gelatine, yet I have there stated, that I did not think it right to conclude, that similar decoctions made under certain circumstances, might not occasionally possess those properties which characterize the tanning substances. Moreover I also observed in the same Paper, that all of those decoctions, upon the addition of a small portion of nitric acid and subsequent evaporation, became converted into that variety of tanning matter which is produced by the action of nitric

acid upon carbonaceous substances. I have since extended these experiments, and shall here give some account of them.

1.

200 grains of the fresh peels of horse chesnuts were digested for about 12 hours in three ounces of distilled water. The liquor was of a pale brown, and formed a slight pale brown precipitate when solution of isinglass was added to it.

2.

200 grains of the same peels were moderately roasted, and being afterwards digested with three ounces of water, formed a dark brown decoction, which was not rendered turbid by gelatine.

3.

The above mentioned roasted peels, after the termination of the preceding experiment, were added to the remainder of the filtrated liquor. A quarter of an ounce of nitric acid was poured upon the whole, which was then digested and evaporated to dryness. The mass was afterwards infused in water, and a dark reddish-brown liquid was obtained, which copiously precipitated solution of isinglass.

4.

200 grains of horse chesnuts, from which the peels employed in the former experiments had been taken, were bruised, and were digested with three ounces of water. The liquor was turbid, and of a pale red colour. It was filtrated, and some solution of isinglass was added, but without any effect.

5.

200 grains of the same horse chesnuts were moderately roasted, and being treated as above described with water, yielded a dark brown decoction which was not rendered turbid by isinglass.

6.

The horse chesnuts, which had been employed in the preceding experiment with the remaining liquor, were digested with a quarter of an ounce of nitric acid until the whole was become dry. Water was then poured upon it, was digested, and a dark brown liquid was formed, which afforded a considerable precipitate by the addition of solution of isinglass.

From these experiments it appears, that the small portion of tannin which the horse chesnut peels originally contained, was destroyed by the process of roasting; that the brown decoction subsequently obtained from the roasted peels and from the horse chesnuts, did not act upon gelatine; but that these were speedily converted into the artificial tanning substance, by the addition of a small portion of nitric acid and subsequent evaporation.

The first preparations of the artificial tanning substance which have been mentioned in the former Papers, were made from coal of different descriptions digested with nitric acid, and as similar products have been obtained by the same acid from various decoctions of roasted vegetable substances, there cannot be any doubt, that vegetable bodies when roasted, yield solutions by digestion in water, which essentially consist of carbon approaching to the state of coal, although not absolutely converted into it, for if so, all solubility in water would cease.

But coal is apparently nothing more than carbon oxidized

to a certain degree, and may be formed by the humid as well as by the dry way.

Examples have been already stated respecting operations in which sulphuric acid has produced this effect, but the same likewise appears to be produced with some modifications, whenever vegetable matter undergoes the putrefactive process; for when this takes place, as in dunghills, &c. a large proportion of the carbon of the original vegetable substances appears to be combined with oxygen sufficient to communicate to it many of the properties of coal, whilst the compound nevertheless is capable of being dissolved by water with the most perfect facility.

It must not however be understood that by this process all the other elementary principles are separated, so that only the carbon remains combined with oxygen, but merely, that the other principles are so far diminished, that these, namely, carbon and oxygen, predominate in a state approaching to coal, although soluble in water.

Such solutions, I have every reason to believe, are nearly similar to those afforded by vegetable substances which have been previously roasted, and although I have examined but a few of them, yet I shall relate some experiments which I have lately made on the peels of walnuts.

It is well known that when these are kept in small heaps for a short time, they become soft, and break down into a black mass, which affords a brownish-black liquor. On these I therefore made the following experiments:

1.

About one ounce of walnut peels, which were become soft and black, was digested in water.

A dark brown liquor was thus formed, and being filtrated,

was examined by a solution of isinglass, but not any apparent effect was produced.

2.

On an equal quantity of the walnut peels, in the same soft black state, a small portion of nitric acid was poured, and after being digested for about five hours, the whole was evaporated to dryness. The residuum was of a brownish orange colour, and yielded a similar coloured solution to water when digested with it. This was filtrated, and upon the addition of solution of isinglass, became turbid, and deposited a tough precipitate, which was not dissolved by boiling water.

3.

Another portion of the walnut peels was moderately roasted, and was then digested in water; the brown solution was filtrated, and formed a slight precipitate with gelatine.

4.

On the residuum of the last experiment, a small quantity of nitric acid was poured, some water was then added, the whole was digested during about five hours, and until it became perfectly dry.

Water formed with this a brown liquor, which yielded a very abundant precipitate by the addition of dissolved isinglass.

Upon these experiments we may remark, that the solution in the first instance contained carbon in a state approaching to coal, for when treated with nitric acid in the second experiment, a portion (although small) was produced of the same tanning substance which is formed from the different kinds of coal by nitric acid.

The third experiment appears to shew, that a small quantity of a substance approaching to tannin was produced by the simple process of roasting; and the fourth experiment corroborates those already described, in which, the artificial tanning matter was copiously produced, whenever roasted vegetable substances were treated with nitric acid.

In respect to vegetable substances, especially those which contain tannin, I shall here relate a few other experiments.

It has been remarked in my second Paper, (p. 288,) that the tannin of galls was immediately destroyed by nitric acid. Since that time, I have made the following additional experiments:

1.

100 grains of galls reduced to powder were infused with four ounces of water, and part of the infusion upon the addition of solution of isinglass afforded (as usual) a copious precipitate of a brownish-white colour.

A quarter of an ounce of nitric acid was added to one ounce of the above infusion, which then, was not in any manner affected by the dissolved isinglass.

2.

100 grains of the same galls were slightly roasted, and being digested with four ounces of water, formed a brown liquor, which was filtrated.

Solution of isinglass was then added to a part of the above liquor, and produced a precipitate not very unlike the former, but much less in quantity.

After this, a quarter of an ounce of nitric acid was added to one ounce of the same liquor, and some dissolved isinglass

was subsequently poured into it, by which it was rendered turbid, and a small portion of a dark brown precipitate was produced, resembling that which is commonly afforded by the artificial tanning substance.

3.

The remainder of the above mentioned liquor, with the residuum of the roasted galls, were digested with a quarter of an ounce of nitric acid until the whole had become dry. Water was then poured upon it, and formed a dark brown solution, which yielded a copious brown precipitate by the addition of dissolved isinglass.

From these experiments on galls it appears, that the natural tannin contained in them is destroyed by nitric acid; that the tannin is also diminished, (and I may add,) is ultimately destroyed by the process of roasting; that when galls have not been so far roasted as to destroy the whole of the tannin, then the remainder of this seems to be destroyed by the addition of nitric acid, whilst at the same time a small portion of the artificial tanning substance is produced; and that this last is always plentifully afforded by roasted galls when digested with nitric acid, similar to other vegetable bodies when thus treated.

These remarks are also partly confirmed by the following experiments upon oak bark.

1.

200 grains of oak bark, reduced into very small fragments, were infused in about four ounces of water, after which the infusion was examined by dissolved isinglass, and yielded a considerable precipitate.

2.

200 grains of the same sort of bark were slightly roasted, and afterwards digested in water; a much darker coloured liquor was obtained than in the former case; but although it afforded precipitates by the addition of muriate of tin, acetite of lead, and sulphate of iron, yet not the smallest effect was produced by solution of isinglass.

3.

The residuum, with the remaining part of the above mentioned liquor, was then digested with a small portion of nitric acid; this was completely evaporated, and a brown solution was formed by water, which abundantly precipitated gelatine.

4.

One ounce of oak bark, reduced into very small fragments, was repeatedly digested in different portions of water until the whole of its tannin was extracted. The residuum or exhausted bark (as it is called by the tanners) was dried, and was afterwards moderately roasted. It was then moistened with diluted nitric acid, which was evaporated in a heat not much exceeding 300° until the bark was become perfectly dry. This was digested in water, and speedily formed a yellowish-brown liquor, which abundantly precipitated gelatine.

5.

The bark, which after being exhausted of its natural tannin, had thus afforded the artificial tanning substance, was repeatedly treated with water until the whole of this last was extracted. The bark was then again slightly roasted, was again

moistened with nitric acid, and was gently heated and dried as before. Water being poured on it and digested, formed a brown solution, which copiously precipitated gelatine.

6.

The whole of the artificial tanning substance was extracted by different portions of water, and the remainder of the bark thus exhausted, was again treated in the manner above described, and again afforded a considerable quantity of the tanning substance, so that these processes evidently might have been continued until the whole of the bark had been converted into it.

This might also have been accomplished, if in the first instance, the exhausted bark had been converted into charcoal, and digested in nitric acid, as described in my first Paper; but then, the effects would have been more slowly produced, and much more nitric acid would have been consumed. I am now therefore fully convinced, not only by the results of the experiments related in this Paper, but also by many others which it would have been superfluous to have stated, that the most speedy and most economical of all the processes which I have described, is that of treating roasted vegetable substances in the way which has been mentioned, and considering that all refuse vegetable matter may be thus converted into a tanning substance by means the most simple, and without any expensive apparatus, I cannot help entertaining much hope, that eventually this discovery will be productive of some real public advantage.

§ V.

In my first Paper I have remarked, that I suspected the tannin of the peat moors to have been produced during the imperfect carbonization of the original vegetable substances. Whether this has been the case, or whether the tannin has at times been afforded by heath and other vegetables growing upon or near the peat, still appears to me to be uncertain; but whatever may be the origin, I never have yet been able to detect any tanning substance in peat, although I have examined a considerable number of varieties, some from Berkshire, and many from Lancashire, which were obligingly sent to me for this purpose by my friend JOHN WALKER, Esq. F. R. S. Mr. JAMESON has also made the same observation,* so that there cannot be any doubt (whatever the origin of the tanning matter may have been) that it has speedily been extracted and drained from the substances which at first contained it.

This effect is a natural consequence of the great facility with which tannin is dissolved by water, and extends even to the most solid vegetable bodies; I shall here give an example.

In the Philosophical Transactions for 1799, Dr. CORREA DE SERRA has given an account of a submarine forest at Sutton, on the coast of Lincolnshire, where submerged vegetables are found in great abundance, including trees of different descriptions, especially birch, fir, and oak. At the time when I was engaged in those experiments on the Bovey coal, and other substances of a similar nature, which have been printed in the Philosophical Transactions for 1804, Sir JOSEPH BANKS

* An Outline of the Mineralogy of the Shetland Islands, &c. 8vo. edition, p. 174.

had the goodness to send me a piece of the oak, which was perfect in all of its vegetable characters, and did not appear to have suffered any change excepting, that it was harder, and of a darker colour than recent oak wood. From some experiments which I then made, I found, that after incineration it afforded potash, similar to the recent wood, and contrary to substances like the Bovey coal, which retain the vegetable external characters, although imperfectly converted into coal.*

In the course of my experiments on tannin, I reduced about an ounce of this submerged oak into shavings, and digested them in water. A brown decoction was formed, which with muriate of tin afforded a pale brown precipitate; with acetite of lead, a precipitate of a deeper brown; with sulphate of iron, a copious brownish-black precipitate; but with solution of isinglass not any effect was produced.

The tannin of this oak wood, had therefore either been separated by solution, or had been decomposed; so that the only substance which remained capable of being dissolved by water, was the extractive matter. This last, in the present case, was most probably the original extractive matter of the oak, but in some other instances, (such, for example, as that which was found in the alder leaves contained in the Iceland schistus,†) I am much inclined to believe, that an extractive substance of secondary formation, if I may be permitted to employ such a term, is produced during the process of carbonization. If a substance, therefore, so compact and solid as oak timber can by long submersion be deprived of its tannin, it naturally follows that the same effect must be more speedily produced by the action of water on the smaller vegetable

* Phil. Trans. for 1804, p. 399. † Ibid. p. 391.

bodies, which present an extensive surface, and also on porous and bibulous substances such as peat.

But although peat, as I have already observed, does not contain any tannin, yet the imperfect carbonization which it has undergone, renders it like the roasted ligneous bodies, peculiarly susceptible of being converted into the artificial tanning substance when exposed to the action of nitric acid. It would be useless to enter into a detail of the different experiments which I have made upon it, as they were similar to those already related, and I shall therefore only here state, that when seven ounces of well dried peat had been twice moistened, and digested with diluted nitric acid, (to the amount of rather more than two ounces,) and subsequently dried, I obtained by water a solution of the artificial tanning substance, which when evaporated to dryness weighed two ounces. I am convinced, that much more might have been obtained from the residuum of the peat, had I thought proper to have repeated the operation; and I am also certain, that less nitric acid would have been sufficient, had the process been conducted in close vessels, and with other economical precautions, which at that time, were for the sake of expedition and convenience omitted.

§ VI.

It has been generally stated, even by modern chemists, that the acids act but little, if at all, upon resinous substances.

The contrary has however been proved, not only in the three Papers upon the present subject, but also in some others which I have formerly had the honour to lay before this learned Society.

In my experiments on lac, printed in the Phil. Trans. for 1804, p. 208, I have particularly endeavoured to shew, how powerfully the acetic acid acts upon resin, gluten, and some other substances; so that it may justly be regarded, as a valuable agent in the chemical analysis of vegetable bodies. In this point of view, it is as a solvent to be the more highly appreciated, because it appears to dissolve the resins, &c. without affecting their respective qualities, and thus by proper precipitants, these substances may be separated from it pure and unaltered.

I am induced therefore to consider acetic acid to be the true acid solvent of the resinous substances, as it dissolves them speedily, without producing any apparent subsequent change in their natural properties.

Sulphuric acid also, almost immediately dissolves the resins, balsams, &c. and forms transparent brown or sometimes crimson solutions, the latter colour being most commonly characteristic of the balsams.

These solutions, however, are different from those made in the acetic acid, by not being permanent, for from the moment when the solution is completed, progressive alterations appear to be produced in the body which is dissolved; thus turpentine is almost immediately converted into resin, then into the third variety of the tanning substance, and lastly into coal.

Without being under the necessity of adducing other examples, we may therefore state sulphuric acid to be a solvent of the resinous substances, but which continues afterwards to act on their principles, so as to decompose them, coal being the ultimate product.

Nitric acid, as I have shewn in the course of these Papers,

and likewise on some former occasions, dissolves the resins, but the progress of its effects seems to be, conversely, that of sulphuric acid; in the latter case, solution precedes decomposition; but when nitric acid is employed, decomposition to a certain degree precedes solution; for it at first converts the resins into a pale orange-coloured brittle, porous substance; then into a product, which apparently possesses the intermediate characters of vegetable extractive matter and of resin, and lastly, this is converted into the first variety of the tanning substance, beyond which I have not been able to effect any change.

As coal therefore appears to be the ultimate effect produced by sulphuric acid upon the resinous bodies, so does the first variety of the tanning substance seem to be the terminating product afforded by the same when acted upon by nitric acid. This effect of nitric acid has been already amply discussed, neither does it appear necessary that I should here repeat the remarks which have been made on some of the simultaneous products, such as the vegetable acids; but amongst the effects produced by sulphuric acid, the coal which is formed seems to merit some attention.

§ VII.

After the tanning substance and the other products had been obtained from the resins, balsams, &c. which have been mentioned in the beginning of this Paper, the following proportions of coal remained.*

* The weight of the coal obtained from each of the above mentioned substances, was estimated after the complete separation of every other product, and after the moisture had been expelled by a red heat, in close vessels.

		Coal.
100 grains of	Copal	67 grains.
———	Mastich	66
———	Balsam of Peru	64
———	Elemi	63
———	Tacamahac	62
———	Guaiacum	58
———	Gum ammoniac	58
———	Amber	56
———	Olive oil	55
———	Balsam of Tolu	54
———	Asa fœtida	51
———	Wax	50
———	Dragon's blood	48
———	Benzoin	48
———	Olibanum	44
———	Myrrh	40
———	Asphaltum	40
———	Gamboge	34
———	Elastic bitumen	31
———	Gum arabic	29
———	Liquorice	25
———	Manna	25
———	Tragacanth	22
———	Caoutchouc	12*

The coal obtained from the resinous bodies by means of sulphuric acid, is in a much greater proportion, than when

* Caoutchouc and elastic bitumen were only superficially carbonized by the sulphuric acid, so that the proportion of coal as above stated, is considerably less than that, which in reality might have been obtained from them.

equal quantities of those substances are exposed to simple distillation.

For, (as I have stated in my first Paper,) 100 grains of common resin by the humid process afforded 43 of coal, which after a red heat still weighed 30 grains.

But the same quantity of resin by distillation, only yielded $\frac{1}{4}$ of a grain of coal.

100 grains of mastich, by the first method, afforded 66 grains of coal.

100 grains of the same mastich only gave $4\frac{1}{2}$ grains of coal when simply distilled.

And 100 grains of amber, when treated with sulphuric acid, yielded 56 grains of coal.

But from 100 grains of the same amber when distilled, only $3\frac{1}{2}$ grains could be obtained.

Many other examples might be adduced, but these appear to be sufficient; and I must here observe, that the case is very different in respect to the gums, for the difference between the proportions of coal obtained from them by the humid and dry ways is not very considerable, although it is always the greatest in the former process, when conducted with precaution. Moreover it is to be remarked, that in either process, variations in the quantity of coal are produced by difference of temperature, by the figure and size of the vessels, and many other circumstances.

But it is not only in the proportion, that there is so great a difference between the coal obtained from the resinous substances by the humid way or by fire, for the quality is also most commonly different; and this not only applies to the resins but also to ligneous matter.

The coal obtained by the humid process from many of the resins, was shining, hard, and occasionally iridescent. Few of the coals obtained from the same bodies by fire had any of these properties. The combustion of the former was slow in the manner of some of the mineral coals, whilst on the contrary the latter were speedily consumed like charcoal. This difference I was at first inclined to attribute to a small portion of the acid which might not have been completely separated, and I therefore purposely made some experiments which convinced me that this was not the case.

Having remarked this difference in the coals afforded by the resins, I was desirous to make some comparative experiments on wood, and for this purpose I selected oak.

1.

On 480 grains of oak sawdust I poured two ounces of sulphuric acid diluted with six ounces of water, and placed the matrass on a sand-bath, where it remained from the beginning of last June to the end of September. During this time, the sand-bath had very seldom been heated, but the vessel was occasionally shaken.

At the end of the period above mentioned, six ounces of boiling water were added, and the whole being poured upon a filter, was repeatedly washed, and was afterwards dried on a sand-bath in a heat not much exceeding 300°.

The sawdust appeared to be reduced to a granulated coal, partly pulverulent, and partly clotted; the whole weighed 210 grains.

105 grains of this coal were put into a platina crucible, and were exposed to a red heat under a muffle. At the same

time, an equal quantity of charcoal made from the same oak sawdust, was placed in another vessel by the side of the former.

The charcoal was speedily consumed, and left some brownish-white ashes, which as usual, afforded alkali, with a trace of a sulphate, which was probably sulphate of potash.

On the contrary, the coal formed by the humid way, burned without flame, similar to the Kilkenny coal, and others which do not contain bitumen. It was very slowly consumed, like the mineral coals above mentioned, and left some pale red ashes, which weighed 2 grains. These ashes *did not yield* the smallest vestige of *alkali*, and the only saline substance which could be obtained, was a very small portion of sulphate of potash, which did not amount to more than $\frac{1}{5}$ of a grain; and it is probable, that had the coal been more copiously washed, even this small portion of the neutral salt would not have been obtained.

2.

At the time when the preceding experiment was began, I also put 480 grains of the oak sawdust into another matrass, and having added four ounces of common muriatic acid, the whole was suffered to remain during the period which has been mentioned.

At the end of the four months, the remainder of the acid was for the greater part driven off by heat not exceeding 300°. The sawdust then had the appearance of a brownish-black mass, on which about a pint of boiling distilled water was poured; the whole was decanted into a filter, was repeatedly washed, and was afterwards dried without heat. The sawdust then appeared, as I have observed, brownish-

black, and was pulverulent. It burned with some flame, emitted still a slight vegetable odour, and was reduced to ashes much sooner than the coal formed by sulphuric acid, but not so speedily as the oak charcoal. The ashes had an ochraceous appearance, and were almost devoid of any saline substance, excepting a very slight trace of muriate of potash.

These two experiments therefore prove,

1st. That wood may by sulphuric acid be converted into a coal which in its properties is very different from charcoal, although prepared from the same sort of wood; and that the coal thus formed by the action of sulphuric acid, resembles by its mode of burning, and by not affording any alkali when reduced to ashes, those mineral coals which are devoid of bitumen.

2dly. That wood may also be converted into a sort of coal by muriatic acid, but in this case some of the vegetable characters remain, although, like the former, not any alkali can be obtained from the ashes.

§ VIII.

Four different solutions have been proposed respecting that difficult problem in the natural history of minerals, *the origin and formation of coal.*

The first is, that pit-coal is an earth or stone chiefly of the argillaceous genus, penetrated and impregnated with bitumen.

But Mr. Kirwan very justly remarks, that the insufficiency of this solution is demonstrated by Kilkenny and other coals which are devoid of bitumen, and also that the quantity of earthy or stony matter in the most bituminous coals bears no proportion to the weight of them.*

* Geological Essays, p. 316.

The second and most prevailing opinion is, that mineral coal is of vegetable origin, that the vegetable bodies have, subsequent to their being buried under vast strata of earth, been mineralized by some unknown process, of which, sulphuric acid has probably been the principal agent, and that by means of this acid, the oils of the different species of wood have been converted into bitumen, and a coaly substance has been formed.

The third opinion is that of ARDUINO; who conceives coal to be entirely of marine formation, and to have originated from the fat and unctuous matter of the numerous tribes of animals that inhabit the ocean.

And the fourth is Mr. KIRWAN's opinion, who considers coal and bitumen to have been derived from the primordial chaotic fluid.*

The limits of this Paper will not permit me to enter into the various arguments and facts which have been adduced in the support of these different opinions; but the second, or that which regards the vegetable substances as the principal origin of coal, seems by much the most probable, because it is corroborated by the greater number of geological facts, as well as by many experimental results. Most of the former have however been stated in different works, and I shall therefore only notice a few of the latter which have occurred in the course of my experiments.

The observations of Dr. CORREA DE SERRA on the wood of the submarine forest at Sutton, on the coast of Lincolnshire, together with many similar accounts which have been published in the Philosophical Transactions and other works, demonstrate in the most satisfactory manner, that whether

* Geological Essays, p. 327.

vegetables are totally or partially buried under the waves or under the earth, they are not merely by such means converted even into the most imperfect sort of coal.* Some process therefore independent of these circumstances must have taken place, in order that the vegetable substances, such as ligneous matter, resin, oil, &c. should become coal and bitumen.

In a former Paper I have endeavoured to shew, that these changes are progressive, and having noticed the perfect state of the submerged wood at Sutton and other places, I next described the qualities of the different kinds of Bovey coal, which exhibit a series of gradual changes from bodies which retain the vegetable structure and texture, although imperfectly carbonized, to others in which almost the complete characters of the common mineral or pit-coal are absolutely established.

From the alder leaves in the schistus from Iceland, I obtained extractive vegetable matter, and although this was not

* In my Paper, "*On the Change of some of the proximate Principles of Vegetables into Bitumen*," I have quoted the remarks of BERGMAN, VON TROIL, and others, on the compressed state of the trunks of the trees which have been converted into surturbrand, Bovey coal, and similar substances. The same observation has been also made by Dr. CORREA DE SERRA respecting the timber of the submarine forest at Sutton; and this is the more remarkable, as the submerged vegetables at Sutton do not exhibit any appearance of carbonization.

Dr. CORREA says, "In general the trunks, branches, and roots of the decayed "trees, were *considerably flattened*; which is a phenomenon observed in the surtur- "brand or fossil wood of Iceland, and which SCHEUCHZER remarked also in the "fossil wood found in the neighbourhood of the lake of Thun, in Switzerland." Phil. Trans. 1799. p. 147.

afforded by the varieties of Bovey coal, yet these, as well as the alder leaves, and also a coal like that of Bovey, found in Sussex, at Newick Park, (an estate belonging to Sir ELIJAH IMPEY,) and also the surturbrand of Iceland, yielded some resin, which at Bovey is likewise found in distinct masses, intermixed with the strata of coal, and combined with asphaltum, in the proportion of about 41 parts of the latter with 55 of resin.*

Now, exclusive of the other vegetable characters which are so evident in many of the varieties of Bovey coal, of the Sussex coal, of surturbrand, &c. &c. the presence of resin must be regarded as a strong fact; for this substance has always been attributed to the organized bodies, particularly to those of the vegetable kingdom, and I do not know of any instance, previous to my own experiments, in which, resin had been discovered as constituting part of any of the different species and varieties of coal.

From the external vegetable characters possessed by the Bovey coal, the Sussex coal, the surturbrand, and many others, together with the resin, (allowed to be exclusively a vegetable substance, or at least one which only appertains to the organized natural bodies,) there cannot be any doubt, that such coals have been formed from wood and other substances belonging to the vegetable kingdom.

But some mineralogists attempt to draw a line of separation between the coals above mentioned and the others, which therefore they call the true mineral coals.

* Observations on the Change of some of the proximate Principles of Vegetables into Bitumen. Phil. Trans. 1804, p. 405.

This opinion may in some degree be refuted even from the specimens afforded by the Bovey coal-pits, where, as I have observed, a regular gradation may be seen from wood which is but very imperfectly carbonized, to the substance called stone coal, which in every respect appears to be most nearly if not absolutely similar to the common pit-coals.*

It may however be objected, that such a transition is peculiar to this and similar places, and that the pit-coal found in other situations, where nothing resembling the Bovey coal can be discovered, is in reality of a different nature.

But this objection I think may be answered by the results of those experiments on pit-coal, Cannel coal, and asphaltum, which I have related in the third section of this Paper; for when these were subjected to the action of nitric acid not too long continued, it was found, that the acid first dissolved the principal part of the carbonaceous matter, and if then the process was stopped, there remained a substance in a proportion corresponding to that of the bitumen either in the pit-coal, or principally forming the Cannel coal and asphaltum, which although not absolutely in the state of resin, was however in a state intermediate between it and the vegetable extractive matter.

Moreover I have stated, that under similar circumstances, a substance possessing in a great measure the same properties, may be obtained from the known vegetable resins by the action of nitric acid.

When therefore, these facts are added to that of the

* Phil. Trans. 1804, p. 398.

natural mixture of resin and asphaltum which is found with the Bovey coal, we to all appearance have almost positive proof that the pit-coals are of vegetable origin.

True it is indeed, that bitumen has never been formed by any artificial process hitherto devised, from the resins or other vegetable substances. I have myself attempted it in various ways without success, for although I occasionally obtained products which resembled it somewhat in odour when burned, and other properties, yet the effects of alcohol or water always proved these products not to be bitumen.

But synthesis of natural products, although required in strict chemical demonstration, is (as we have but too often occasion to know) seldom to be attained, especially when operations are performed on bodies whose component parts are liable to an infinite series of variations in their proportions, qualities, and mode of combination.

Considering therefore, that bitumen and resin afford by certain operations similar products, that resin and bitumen are found blended together by nature, and that this mixed substance accompanies a species of coal which in many parts still exhibits its vegetable origin, whilst in others it passes into pit-coal, we may with the greatest probability conclude, that bitumen is a modification of the resinous and oily parts of vegetables, produced by some process of nature, which has operated by slow and gradual means on immense masses, so that even if we were acquainted with the process, we should scarcely be able to imitate its effects, from the want of time, and deficiency in the bulk of materials.

But although bitumen cannot at present be artificially

formed from the resinous and other vegetable substances by any of the known chemical processes, yet there is every reason to believe, that the agent employed by nature in the formation of coal and bitumen has been either muriatic or sulphuric acid; and when it is considered, that common salt is never found in coal mines except when in the vicinity of salt springs, whilst on the contrary, pyrites, sulphate of iron, and alum, most commonly are present;* these facts, together with the sulphureous odour emitted by most of the mineral coals when burned, appear strongly to evince the agency of the latter. That this has been the case, seems also to be corroborated, by the great resemblance which (as has been previously stated) the coals formed artificially from many vegetable substances bear to the mineral coals, especially as the similarity is not confined to external characters, but extends to other properties.

By the action of sulphuric acid on vegetable bodies, a much greater portion of their carbon is converted into coal than when the same are subjected to the effects of fire.

Several examples respecting the resins, have been mentioned in the seventh section of this Paper, and the result of the experiment made upon oak perfectly accords with them.

Mr. Proust, in the course of some comparative experiments on the proportions of charcoal afforded by different kinds of wood, obtained 20 *per cent.* from green oak, and 19 *per cent.* from heart of oak.†

* Kirwan's Geological Essays, p. 324.

† *Journal de Physique*, 1799, Tome 48, p. 469.

But by sulphuric acid, from 480 grains of oak, I obtained 210 grains, or about 45 *per cent.* of coal, which burned not like the charcoal obtained from the same wood, but like many of the mineral coals; and this was also observed in the combustion of the greater part of the coals obtained by the humid way from resinous substances.

The experiment on oak also appears to refute another objection to the vegetable origin of pit-coal, namely, the total absence of the alkalis, which on the contrary are so constantly obtained from the ligneous parts of vegetables by combustion.* But I have shewn, that when these bodies are carbonized in the humid way either by muriatic or by sulphuric acid, not any alkali can be obtained from the ashes of coals so formed; and this seems also to be a farther proof, that the humid way has been employed in the operations of nature to convert the above mentioned substances into pit-coal; for supposing fire to have been the agent, it does not appear easy to conceive how the alkali could have been destroyed or separated.†

* KIRWAN's Geological Essays, p. 320.

† Some have attempted to account for the absence of alkali in the Bovey coal and common pit-coal, by supposing that the vegetable bodies (from which these have been formed) were previously deprived of alkali by simple lixiviation during their immersion in water. But in page 127 of this Paper, I have shewn that the submerged oak of Sutton, although deprived of its tannin, still retained its potash, which certainly would not have been the case if the latter like the former could have been separated from the wood by mere solution. When wood is reduced to ashes, the alkali becomes completely denuded by the destruction of the woody fibre, and consequently may be immediately taken up by water; but when wood is converted into coal in the humid way by means of an acid, then it seems to me that two effects

Every circumstance seems therefore to support the opinion of those who consider the pit-coals as having been formed in the humid way, principally from vegetable bodies, and most probably by the agency of sulphuric acid; and allowing that animal substances may also have contributed to the production of coal, yet this would not militate against the above mentioned opinion, as the effects produced upon them by that acid would in all the essential points be perfectly similar.*

An inquiry into the nature and formation of coal was my first object when I discovered the artificial tanning substance,

take place; for the intimate combination of the alkali with the woody fibre becomes in a great measure destroyed by the carbonization of the latter, whilst a simultaneous action arises in the affinity between the acid and the alkali; so that if coal has been formed by such means, the alkali must have been separated from the wood in the state of a dissolved neutral salt.

* From the nature of the experiments which have been related in this Paper, I have unavoidably been induced to notice concisely the different opinions on the formation of coal by the humid way; but I did not intend to have mentioned any of those which have been brought forward in favour of the immediate or indirect action of fire, as I only wished to express my sentiments respecting the most probable of the former opinions.

Since however this Paper was written and partly read before the Royal Society, I have been favoured by Sir James Hall, with a copy of his Paper, intitled "*Account of a Series of Experiments shewing the Effects of Compression in modifying the Action of Heat*;" and I am fully of opinion that the scientific world has not for a long time received any communication of more importance, or in which more accuracy, ability, and perseverance have been displayed. The effects which Sir James Hall has produced on carbonate of lime by heat acting under compression, certainly removes a great and at one time apparently insurmountable obstacle to the Huttonian or Plutonian theory, and if they do

and considering the importance of the latter, it will not appear surprising, that it should immediately have engaged the principal part of my attention.

In addition to the experiments which have been related in the three Papers upon this subject, I intended to have decomposed the different varieties, to have compared their gases and other products with those of the natural substance called Tannin, and especially to have endeavoured to discover more economical methods of obtaining the artificial product; for, exclusive of speculative science, this appears to be an object of consequence, not only respecting that useful and valuable branch of manufacture to which it immediately relates, but also as the means of preventing, or at least of diminishing, the premature destruction of timber in a country, where, on account of its population, as well as on account of its maritime

not solve the grand geological problem, they must even, in an insulated point of view, be allowed to have opened a new and unexplored field of research in chemistry as well as in geology.

In the 8th section of this valuable Paper, the author has given an account of some experiments made on leather, horn, and fir sawdust, from which he obtained coal which burned with flame, and which apparently resembled some of the mineral coals. In one case also, he obtained a substance, which in external characters appeared somewhat similar to the mixture of asphaltum and resin found at Bovey, to which I have given the name of Retin-asphaltum. These experiments Sir JAMES HALL intends to resume, and it is my earnest wish that he would do so; for although I am strongly inclined to believe that the mineral coals have generally, if not always, been formed by some humid process, yet it is impossible to foresee the results which may be obtained from animal and vegetable bodies subjected to the effects of heat modified by compression, as the principles of these bodies may be acted upon, and may be made to re-act on each other, under circumstances which until now have not been imagined.

position, every economy in such an article should be most rigidly observed.

But for the present, I intend to relinquish this subject to such as may consider it worthy of attention; whilst, as I have already stated, I entertain very sanguine expectations, that eventually it will prove economically useful; and should any be inclined to pursue the inquiry, I would recommend particular attention to those processes which relate to the roasted vegetable substances, and to peat.

Almost any refuse vegetable matter, such as twigs, dead leaves, &c. will serve for the former; whilst the latter, as I have shewn, does not require to be roasted, and in many, especially the northern counties, peat is found in such abundance, that but a small proportional quantity is consumed in the only useful way to which it has hitherto been applied, namely, fuel.

Before I conclude this Paper I shall also observe, that the experiments which have been described, must be regarded only as a mere sketch of that which may be performed, whilst the facts which have been ascertained respecting the resins, balsams, gum resins, and gums, serve to prove, that much may be expected from regular chemical examinations of these bodies. But such investigations, in order that science may truly be promoted, should be strictly regular: that is, they should not be taken up in a desultory manner, but these substances should be comparatively and systematically examined with all the accuracy which can be employed in the present state of chemical knowledge; for as this knowledge concerning the composition

of organized bodies is confessedly very imperfect, I am persuaded, that like other of the sciences, chemistry will be less liable to error, when guided by comparative experiments, and comparative analyses.

VII. *The Application of a Method of Differences to the Species of Series whose Sums are obtained by Mr.* Landen, *by the Help of impossible Quantities. By Mr.* Benjamin Gompertz. *Communicated by the Rev.* Nevil Maskelyne, *D. D. Astronomer Royal, F. R. S.*

Read February 13, 1806.

HAVING some years back, when reading the learned Mr. LANDEN's fifth Memoir, discovered the manner of applying a method of differences, to the species of series, whose sums are there obtained by the help of impossible quantities, and having since extended that application, I now venture to offer it to the consideration of others.

The practice of this method, in most cases, appears to me extremely simple; and on that account, I am almost induced to imagine, that they have already been considered by mathematicians; indeed since the greatest part of this Paper was written, I met with EULER's *Institutiones Calculi integralis*; two simple series are in that work summed by multiplications similar to those employed in the investigation of the principal theorems contained in this Paper; but whether that learned mathematician has farther pursued the method, in that or in any other work, I have not as yet been able to ascertain.

I have purposely considered some of the series summed by Mr. LANDEN, to afford an opportunity of comparing both the results and methods; and because the series may have parti-

cular cases in which both Mr. LANDEN's means and my own fail: I have added towards the end a general scholium concerning the cause, circumstances, and consequences of such failure in my method.

The foundation of the theorems depends on the following well known lemmas.

No. I.

2 sine of vx . sine of tx, is equal to

cos. of $\overline{t-v}\,.\,x$ — cos. of $\overline{t+v}\,.\,x$.

No. II.

2 sine of vx . cos. of tx, is equal to

sine of $\overline{t+v}\,.\,x$ — sine of $\overline{t-v}\,.\,x$, or

sine of $\overline{t+v}\,.\,x$ + sine of $\overline{v-t}\,.\,x$.

No. III.

2 cos. of vx . cos. of tx, is equal to

cos. of $\overline{t-v}\,.\,x$ + cos. of $\overline{t+v}\,.\,x$.

Theorem I.

If there be an infinite series a . sine of px, + b . sine of $\overline{p+q}\,.\,x$, + c . sine of $\overline{p+2q}\,.\,x$, + d . sine of $\overline{p+3q}\,.\,x$ &c. $= s$, and from the series a, b, c, d, e, f, &c. there be continually formed new series $\left\{\begin{matrix} a, a', b', c', d', e', \text{&c.} \\ a, a'', b'', c'', d'', e'', \text{&c.} \\ a, a''', b''', c''', d''', e''', \text{&c.} \\ \text{&c. &c. &c. &c. &c. &c. &c.} \end{matrix}\right\}$

Every new series, being formed from that immediately above, by taking the differences of the terms, exactly in

the same manner as in the common differential method, except that they here continually commence with the first term, a; and if p' be put $= p - \frac{1}{2}q$, $p'' = p' - \frac{1}{2}q$, $p''' = p'' - \frac{1}{2}q$, $p^{iv} = p''' - \frac{1}{2}q$. &c. s . 2 sine of $\frac{1}{2}qz = s'$, $-s'$. 2 sine of $\frac{1}{2}qz = s''$, s'' . 2 sine of $\frac{1}{2}qz = s'''$, $-s'''$. 2 sine of $\frac{1}{2}qz = s^{iv}$ &c. Then shall

a . cos. of $p'z + a'$. cos. of $\overline{p' + q}$. $z + b'$. cos. of $\overline{p' + 2q}$. $z + c'$ cos. of $\overline{p' + 3q}$. $z +$ &c. $= s'$,

a . sine of $p''z + a''$. sine of $\overline{p'' + q}$. $z + b''$. sine of $\overline{p'' + 2q}$. z &c. $= s''$,

a . cos. of $p'''z + a'''$. cos. of $\overline{p''' + q}$. $z + b'''$. cos. of $\overline{p''' + 2q}$. z &c. $= s'''$,

a . sine of $p^{iv}z + a^{iv}$. sine of $\overline{p^{iv} + q}$. $z + b^{iv}$. sine of $\overline{p^{iv} + 2q}$. z &c. $= s^{iv}$,

&c, + &c, + &c, =, &c.

For, multiplying the series a . sine of $pz + b$. sine of $\overline{p + q}$. z $+ c$. sine of $\overline{p + 2q}$. z &c. $= s$, by 2 . sine of $\frac{1}{2}qz$ by lemma No. I. we get, a . cos. of $\overline{p - \frac{1}{2}q}$. $z - a$. cos. of $\overline{p + \frac{1}{2}q}$. $z + b$. cos. of $\overline{p + \frac{1}{2}q}$. $z - b$. cos. of $\overline{p + \frac{1}{2}}$. $qz + c$. cos. of $\overline{p + \frac{1}{4}}$. $qz - c$. cos. of $\overline{p + \frac{5}{2}q}$. z &c. $= s$. 2 sine of $\frac{1}{2}qz$ $\therefore$ putting $b - a = a'$, $c - b = b'$, $d - c = d'$, &c. $p - \frac{1}{2}q = p'$, s . 2 sine of $\frac{1}{2}qz = s'$ we have, a . cos. of $p'z + a'$. cos. of $\overline{p' + q}$. $z + b'$. cos. of $\overline{p' + 2q}$. $z + c'$. cos. of $\overline{p' + 3q}$. z &c. $= s'$, multiply this by 2 sine of $\frac{1}{2}qz$ by help of lemma No. II. and we have $-a$ sine of $\overline{p' - \frac{1}{2}q}$. $z + a$ sine of $\overline{p' + \frac{1}{2}q}$. $z - a'$ sine of $\overline{p' + \frac{1}{2}q}$. $z + a'$ sine of $\overline{p' + \frac{3}{2}q}$. $z - b'$ sine of $\overline{p' + \frac{3}{2}q}$. $z + b'$ sine of $\overline{p' + \frac{5}{2}q}$. $z - c'$ sine of $\overline{p' + \frac{5}{2}q}$. $z + c'$ sine of $\overline{p' + \frac{7}{2}q}$. z &c. $= s'$. 2 sine of $\frac{1}{2}qz$, put $b' - a' = a''$, $c' - b' = b''$, $d' - c' = d''$ &c. $p' - \frac{1}{2}q = p''$ and $-s'$. 2 sine of $\frac{1}{2}qz = s''$, and we have a sine of $p''z + a''$ sine of $\overline{p'' + q}$. $z + b''$ sine of $\overline{p'' + 2q}$. z &c. $= s''$, and because this is exactly similar to the original equation, (if we put a'', b'', c'', &c. for b, c, d, &c. in that, and

p'' and s'' for p and s,) it follows that if we put $b'-a''=a''$, $c''-b''=b''$, $d''-c''=c''$, &c. $p''-\frac{1}{2}q=p''$, s''. 2 sine of $\frac{1}{2}qz=s''$, that we shall have, a cos. of $p''z+a''$ cos. of $\overline{p''+q}.z+b''$, cos. of $\overline{p''+2q}.z$ &c. $=s''$, which is exactly similar to the second equation; (if a''', b''', c''', &c. p''' and s''' be written for a'', b'', c'', &c. p' and s' in that,) and therefore putting $b'''-a'''=a'''$, $c'''-b'''=b'''$, $d'''-c'''=c'''$ &c. $p'''-\frac{1}{2}q=p'''$, $-s'''$. 2 sine of $\frac{1}{2}qz$ $=s'''$, we get a sine of $p'''z+a'''$ sine of $\overline{p'''+q}.z+b'''$. sine of $\overline{p'''+2q}.z$, &c. $=s'''$, again, similar to the first, by putting a'', b'', c'', &c. p'', s'' in that equation for b, c, d, &c. p and s, and thus do we continually get equations in form similar to the first and second equations QED.

Cor. 1. $s''=-s'$. 2 sine of $\frac{1}{2}qz=-s$. 2 sine of $\frac{1}{2}qz|'$, $s'''=s''$. 2 sine of $\frac{1}{2}qz=-s'$. 2 sine of $\frac{1}{2}qz|'=-s$. 2 sine of $\frac{1}{2}qz|'$, $s''=-s'''$. 2 sine of $\frac{1}{2}qz=s'$. 2 sine of $\frac{1}{2}qz|'=s$. 2 sine of $\frac{1}{2}qz|''$, and in general put $s^{(\pi)}$ to represent the πth successive value of s, and we shall have $s^{(\pi)}=\pm s'$. 2 sine of $\frac{1}{2}qz|^{\pi-1}=\pm s$. 2 sine of $\frac{1}{2}qz|^{\pi}$, the upper sign to be taken when π being divided by 4 leaves 0 or 1, the under when it leaves 2 or 3. πth successive value of $p=p-\pi.\frac{1}{2}q$, note the values s', s'', s''', &c. I call successive sums of s, and $s=\pm\frac{s^{(\pi)}}{2 \text{ sine of } \frac{1}{2}qz|^{\pi}}=\pm\frac{s^{(\pi-1)}}{2 \text{ sine of } \frac{1}{2}qz|^{\pi-1}}$.

Corollary II. If A, B, C, &c. A′, B′, C′, &c. A″, B″, C″, &c. &c. be put for the series of the 1st, 2d, 3d differences &c. of the series a, b, c, &c. taken according to the common method of differences, we shall have the series

a, a', b', c', &c. the same as the series a, A, B, C, D, &c.
a, a'', b'', c'', &c. $= a$, a'', A', B', C', &c.
a, a''', b''', c''', &c. $= a$, a''', b'', A'', B'', &c.
a, a^{iv}, b^{iv}, c^{iv}, &c. $= a$, a^{iv}, b^{iv}, c^{iv}, A''', &c.
&c. &c. &c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

This is evident by taking the differences by both methods, and comparing them.

Cor. III. Likewise if A, B, C, &c. A', B', C', &c. A'', B'', C'', &c. &c. be put for the series of the 1st, 2d, 3d, &c. differences of the series a, a', b', c', &c. found by the common method of differences, then shall the series

a, a'', b'', c'', d'', &c. $= a$, A, B, C, &c.
a, a'', b'', c'', d'', &c. $= a$, a'', A', B', &c.
a, a^{iv}, b^{iv}, c^{iv}, d^{iv}, &c. $= a$, a^{iv}, b^{iv}, A'', &c.
&c. &c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

These things being known, we shall now propose some examples of their use.

Example 1. Required the sum of the infinite series sine of $pz+$ sine of $p+q.z+$ sine of $\overline{p+2q}z+$ sine of $\overline{p+3q}.z$ &c.
Here a, b, c, &c. $= 1, 1, 1, 1, 1$, &c. } therefore s' or $s.2$ sine of $\frac{1}{2}qz = \text{cos. of } p'z = \text{cos. of } p-\frac{1}{2}q.z$ $\therefore$ the sum $s = \frac{\text{cos. of } p-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz}$.
a, a', b', &c. $= 1, 0, 0, 0, 0$, &c.

Cor. 1. If p and q were each $=1$, we should have, sine of $z+$ sine of $2z+$ sine of $3z$ &c. $= \frac{\text{cos. of } \frac{1}{2}z}{2 \text{ sine of } \frac{1}{2}z} = \frac{1}{2}$ cotangent of $\frac{1}{2}z$.

Cor. II. If p were $=\frac{1}{2}q$, we should have sine of $pz+$ sine of $3pz+$ sine of $5pz$ &c. $= \frac{\text{cos. of } \frac{1}{2}p.z}{2 \text{ sine of } \frac{1}{2}pz} = \frac{1}{2 \text{ sine of } \frac{1}{2}pz} = \frac{1}{2}$ cosecant of pz.

Example 2. Required the sum of the infinite series, cos. of nz + cos. of $n+qz$ + cos. of $n+2q.z$ &c.

Here writing n in the room of p' we have

$a, a', b', c',$ &c. $= 1, 1, 1, 1,$ &c. $\Big\}$ therefore s'' or $-s'$. 2 sine of $\frac{1}{2}qz =$ sine of $p'z =$ sine of $n-\frac{1}{2}q.z$. s' the sum $= -\frac{\text{sine of } n-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz}$

$a, a', b'', c'',$ &c. $= 1, 0, 0, 0,$ &c. $\Big\}$

Cor. 1. If $n=\frac{1}{2}q$, we shall have cos. of nz + cos. of $3nz$ + cos. of $5nz$ &c. $= -\frac{\text{sine of } n.z}{2 \text{ sine of } \frac{1}{2}qz} = 0$.

Cor. 11. If $n=q$, we shall have cos. of nz + cos. of $2nz$ + cos. of $3nz$ &c. $= -\frac{\text{sine of } \frac{1}{2}qz}{2 \text{ sine of } \frac{1}{2}qz} = -\frac{1}{2}$.

Example 3. Required the sum of the infinite series, sine of nz + 4 sine of $n+q.z$ + 9 sine of $n+2q.z$ + 16 sine of $n+3q.z$ &c. Here $p=n$

$a, b, c, d,$ &c. $= 1, 4, 9, 16, 25,$ &c. $\Big\}$ therefore s'' or $-s$.

$a, a', b', c',$ &c. $= 1, 3, 5, 7, 9,$ &c. $\Big\}$ 2 sine of $\frac{1}{2}qz|^3 =$ cos. of $p''z +$ cos. of $p''+q.z =$ cos. of $n-\frac{1}{2}q.z$

$a, a'', b'', c'',$ &c. $= 1, 2, 2, 2, 2,$ &c. $\Big\}$

$a, a''', b''', c''',$ &c. $= 1, 1, 0, 0, 0,$ &c. $\Big\}$

+ cos of $n-\frac{1}{2}q.z$, and therefore s the sum $= -\frac{\text{cos. of } n-\frac{1}{2}q.z + \text{cos. of } n-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz|^3}$.

Cor. 1. If $n=\frac{1}{2}q$, we have, sine of nz + 4 sine of $3nz$ + 9 sine of $5nz$ &c. $= -\frac{\text{cos. of } -2nz+1}{2 \text{ sine of } nz|^3} = -\frac{\text{cos. of } 2nz+1}{2 \text{ sine of } nz|^3} = -\frac{\text{versed sine supplement of } 2nz}{2 \text{ sine of } nz|^3}$.

Cor. 11. If $n=q$, we shall have, sine of nz + 4 sine of $2nz$ + 9 sine of $3nz$ &c. $= -\frac{\text{cos. of } -\frac{1}{2}nz + \text{cos. of } \frac{1}{2}nz}{2 \text{ sine of } \frac{1}{2}nz|^3} = -\frac{2 \text{ cos. of } \frac{1}{2}nz}{2 \text{ sine of } \frac{1}{2}nz|^3}$, because, cos. of $-nz =$ cos. of $+nz$.

Scholium 1. It is evident from *Cor.* II. and III. *Theorem* I. that if the coefficients of the sines (a, b, c, &c.) or of the cosines (a, a', b', c', &c.) be such that any order of differences taken according to the common method becomes $= 0$, we shall then have the corresponding value, of the successive values of s, s', s'', &c. expressed in finite terms, and we shall consequently get the value of the series sought expressed in finite terms, and likewise all the intermediate values of s', s'', s''', &c. contained between s and the said corresponding successive value of s, expressed in finite terms; hence if the values of a, b, c, &c. or of a, a', b', c', &c. be respectively equal to $\frac{r}{t}g$, $\frac{r \cdot r+b}{t \cdot t+b}g$, $\frac{r \cdot r+b \cdot r+2b}{t \cdot t+b \cdot t+2b}g$, &c. r, h, t, being all affirmative values, and $r-t$ a multiple of h, we may obtain the sum of the series.

In order to prove this, I shall put r, r, r, &c. to represent $r+h$, $r+2h$, $r+3h$, &c. t, t, t, &c. for $t+h$, $t+2h$, $t+3h$, &c. then will the increment of $\frac{r\,r\,r\,\ldots\ldots\,r}{t\,t\,t\,\ldots\ldots\,t}$ $= \frac{r\,r\,r\,\ldots\ldots\,r}{t\,t\,t\,\ldots\ldots\,t\,t+1}$ $-$ $\frac{r\,r\,r\,\ldots\ldots\,r}{t\,t\,t\,\ldots\ldots\,t}$ (ν being supposed the only variable quantity.) $= \frac{r\,r\,r\,\ldots\ldots\,r}{t\,t\,t\,\ldots\ldots\,t} \times \frac{r-t-\nu b}{t+\nu b}$ $= \frac{r\,r\,r\,\ldots\ldots\,r}{t\,t\,t\,\ldots\ldots\,t\,t+\nu b} \times \frac{r-t-\nu b}{1}$; it is likewise evident that the $\nu+1$th term of the series proposed may be expressed by $\frac{r\,r\,r\,\ldots\ldots\,r}{1\,2\,3\,\ldots\ldots\,\nu} \cdot g$, ($\nu$ being a whole positive number,) this term we will call T, therefore we have, from

what has been just shown, $T = \frac{r-i}{i} \cdot g \times \frac{r \ldots r}{i \ldots i}$, $T = \frac{r-i}{i} \cdot \frac{r-i-b}{i+b} \cdot g \times \frac{r \ldots r}{i \ldots i}$,

$\frac{r-i-b}{i+b} \cdot g \times \frac{r \ldots r}{i \ldots i}$, $T = \frac{r-i}{i} \cdot \frac{r-i-b}{i+b} \cdot \frac{r-i-2b}{i+2b} \cdot g \times \frac{r \ldots r}{i \ldots i}$, &c.

and the sth increment or difference $= \frac{r-i}{i} \cdot \frac{r-i-b}{i+b} \cdot \frac{r-i-2b}{i+2b}$ &c.

$\ldots\ldots \times \frac{r-i-(s-1)b}{i+(s-1)b} \cdot g \times \frac{r \ldots r}{i \ldots i}$ which, it is evident, will be equal to 0. If $r-i=(s-1)h$ whatever s may be, that is, whatever term of the sth order of difference be sought it will be found equal to 0; the truth of this will be likewise evinced in particular cases by the following examples.

Example 4. Required the sum of the infinite series, $\frac{1}{1}$ sine of $px + \frac{3}{1.2}$ sine of $\overline{p+q} \cdot x + \frac{3+5}{1.2.3}$ sine of $\overline{p+q} \cdot x$ &c.

Here $a, b, c, d,$ &c. $= 1, \frac{3}{1.2}, \frac{3+5}{1.2.3}, \frac{3+5.6}{1.2.3.4},$ &c.

$a, a', b', c',$ &c. $= 3, \frac{1.3}{1.2}, \frac{2.3.4}{1.2.3}, \frac{3.4.5}{1.2.3.4},$ &c.

$a, a'', b'', c'',$ &c. $= 3, 0, 1, 1,$ &c.

$a, a''', b''', c''',$ &c. $= 3, -3, 1, 0,$ &c.

therefore s'' or $-s.2$ sine of $\frac{1}{2}qx)^3 = 3$ cos. of $\overline{p''x} - 3$ cos. of $\overline{p''+q} \cdot x +$ cos. of $\overline{p''+2q} \cdot x \therefore s$ the sum $= [3$ cos. of $\overline{p''x} -$ 3 cos. of $\overline{p''+q} \cdot x +$ cos. of $\overline{p''+2q} \cdot x] \div -2$ sine of $\frac{1}{2}q \cdot x)^3 =$ $\frac{3 \text{ cos. of } \overline{p-\frac{1}{2}q}\, x - 3 \text{ cos. of } \overline{p-\frac{1}{2}q}\, x + \text{cos. of } \overline{p+\frac{1}{2}q}\, x}{-2 \text{ sine of } \frac{1}{2}qx)^3}$.

Note. The series might have been written thus, 3 sine of $px + 6$ sine of $\overline{p+q} \cdot x + 10$ sine of $\overline{p+2q} \cdot x$ &c.

Example 5. Required the sum of the infinite series, $\frac{1}{3}$ cos. of $nx + \frac{3.6}{3.4}$ cos. of $\overline{n+q} \cdot x + \frac{3.6.7}{3.4.5}$ cos. of $\overline{n+2q} \cdot x$ &c.

Here $p'=n$, therefore,

$$\left.\begin{array}{lll} a, a', b', c', \&c. = \frac{1}{3}, & \frac{1.6}{3.4}, & \frac{1.6.7}{3.4.5}, \quad \frac{1.6.7.8}{3.4.5.6}, \quad \frac{1.6.7.8.9}{3.4.5.6.7}, \quad \&c. \\ a, a'', b'', c'', \&c. = \frac{1}{3}, & \frac{2.5}{3.4}, & \frac{2.5.6}{3.4.5}, \quad \frac{1.5.6.7}{3.4.5.6}, \quad \frac{2.5.6.7.8}{3.4.5.6.7}, \quad \&c. \\ a, a''', b''', c''', \&c. = \frac{1}{3}, & \frac{-2.5}{3.4}, & \frac{2.5}{3.4.5}, \quad \frac{2.5.6}{3.4.5.6}, \quad \frac{2.5.6.7}{3.4.5.6.7}, \quad \&c. \\ a, a^{iv}, b^{iv}, c^{iv}, \&c. = \frac{1}{3}, & \frac{-3}{2}, & 1, \quad 0, \quad 0, \quad \&c. \end{array}\right\}$$

therefore s^{iv} or s'. $2 \sin. \text{of } \frac{1}{2}qz)^3 = \frac{1}{3} \sin. \text{of } p'''z - \frac{1}{2} \sin. \text{of } p''z + q.z + \sin. \text{of } p''+2q.z$, and therefore s' the sum sought

$$= \frac{\frac{1}{3} \sin. \text{of } p'''z - \frac{1}{2} \sin. \text{of } p''+q.z + \sin. \text{of } p''+2q.z}{2 \sin. \text{of } \frac{1}{2}qz)^3} = \left[\frac{2}{3} \sin. \text{of } n - \frac{1}{2}q.z - \frac{1}{2} \sin. \text{of } n - \frac{1}{2}q.z + \sin. \text{of } n + \frac{1}{2}q.z\right] \div 2 \sin. \text{of } \frac{1}{2}qz)^3.$$

Note. The series might have been written thus, $\frac{1.5}{1.2}$ cos. of $nz + \frac{5.6}{6.2}$ cos. of $z + q.z + \frac{6.7}{1.2}$ cos. of $z + 2qz$ &c.

Cor. If $2z = q$, s' becomes $\frac{\frac{1}{3} \sin. \text{of } -3nz - \frac{1}{2} \sin. \text{of } 0 + \sin. \text{of } 2nz}{2 \sin. \text{of } nz)^3} =$

$$\frac{\frac{1}{3} \sin. \text{of } 2nz}{2 \sin. \text{of } nz)^3} = \frac{4}{3} \frac{\sin. \text{of } nz \,..\, \cos. \text{of } nz}{2 \sin. \text{of } nz)^3} = \frac{1}{6} \frac{\cos. \text{of } nz}{\sin. \text{of } nz)^2}, \text{ for the}$$

sum of the series, $\frac{1}{6}$ cos. of $nz + \frac{5.6}{3.4}$ cos. of $3nz + \frac{5.6.7}{3.4.5}$ cos. of $5nz$ &c. or its equal, $\frac{1.1}{1.2}$ cos. of $nz + \frac{5.6}{1.2}$ cos. of $3nz + \frac{6.7}{1.2}$ cos. of $5nz$ &c. $\therefore \frac{-2 \cos. \text{of } nz}{\sin. \text{of } nz)^2} = 1.5 \cos. \text{of } nz + 5.6 \cos. \text{of } 3nz$ &c.

Scholium 11. It is not always necessary for the differences of the coefficients to become equal to 0 to obtain the sum of the series, as will appear by

Example 6. Required the sum of the infinite series sine of $pz+g$ sine of $p+q.z+g^2$ sine of $p+2q.z+g^3$ sine of $p+3q.z$ &c.

We have therefore

$$\left.\begin{array}{l} a, b, c, d, e, \&c. = 1, g, g^3, g^6, g^{10}, \&c. \\ a, a', b', c', d', \&c. = 1, g-1, g\times g-1, g^3\times g-1, g^6\times g-1, \&c. \\ a, a'', b'', c'', d'', \&c. = 1, g-2, \overline{g-1}^2, g.\overline{g-1}^2, g^3.\overline{g-1}^2, \&c. \end{array}\right\}$$

Consequently, s'' or $-s.2$ sine of $\frac{1}{2}qz|^2 =$ sine of $p''z+g-2$. sine of $\overline{p''+q}.z+g-1|^2$. sine of $\overline{p''+2q}.z+g-1|^2.g$ sine of $\overline{p''+3q}.z$ &c. $=$ sine of $\overline{p-q}.z+g-2$. sine of $pz+g-1|^2$. sine of $\overline{p+q}.z+g-1|^2.g$ sine of $\overline{p+2q}.z+g-1|^2.g^3$ sine of $\overline{p+3q}.z$ &c. but, $s=$ sine of $pz+g$ sine of $\overline{p+q}.z+g^3$ sine of $\overline{p+2q}.z$ &c. Consequently, by multiplication, division, and transposition, $g-1|^2$ sine of $\overline{p+q}.z+g-1|^2.g$ sine of $\overline{p+2q}.z+g-1|^2.g^3$ sine of $\overline{p+3q}.z$ &c. $= s.\frac{\overline{g-1}^2}{g} - \frac{\overline{g-1}^2}{g}$. sine of pz, consequently the above equation becomes by substitution s' or $-s.2.$sine of $\frac{1}{2}qz|^2 =$ sine of $\overline{p-q}.z+g-2$ sine of $pz+\frac{\overline{g-1}^2}{g}.s-\frac{\overline{g-1}^2}{g}$. sine of pz, therefore, s the sum required $= \frac{\text{sine of } \overline{p-q}.z+g-2-\overline{g-1}^2.\text{ sine of } pz}{g}$

$$\frac{}{-\overline{g-1}^2-2 \text{ sine of } \frac{1}{2}qz|^2} = \frac{-g \text{ sine of } \overline{p-q}.z+\text{sine of } pz}{g^2+1-2g.\text{ cos. of } qz},$$ and

by similar means, we have the sum of the series, cos. of pz $+g$ cos. of $\overline{p+q}.z+g^2$ cos. of $\overline{p+2q}.z$ &c. $= \frac{-g\text{ cos. of } \overline{p-q}.z+\text{ cos. of } pz}{g^2+1-2g.\text{ cos. of } qz}$.

Scholium III. Hitherto we have been considering, a series of sines and cosines, whose terms have all the same signs; but if the terms of a series proposed were alternately positive and negative, it would be necessary to divide them into two series, the one of the positive term and the other of the nega-

tive; in order to get the sum by *Theorem* I. But the sum of a series whose terms are alternately positive and negative, may be obtained from the sum of a similar series, whose terms are all positive by a mere substitution: thus if the sum of the series, a sine of $rz-b$ sine of $\overline{r+s}.z+c$ sine of $\overline{r+2s}.z-$&c. were required, put $rz=180°-pz$, and $sz=180-qz$, therefore the sine of $rz=$ sine of $180°-pz=$sine of pz, sine of $\overline{r+s}.z=$ sine of $\overline{360°-p+q}.z=-$ sine of $\overline{p+q}.z$, sine of $\overline{r+2s}.z=$sine $\overline{540°-p+2q}.z=$sine of $\overline{p+2q}.z$, &c.; and consequently the sum of the series, a . sine of $rz-b$. sine of $\overline{r+s}.z+c$ sine of $\overline{r+2s}.z-$ &c.= the sum of the series, a sine of $pz+b$ sine of $\overline{p+q}.z+c$ sine of $\overline{p+2q}.z$ &c.; and by the like substitution may the sum of a series of cosines, whose terms are alternately positive and negative, be deduced from the sum of a series of cosines, whose terms are all positive: all this requires the functional values of p and q to be distinct, otherwise the substitution cannot be effected; but the said sum may be deduced at once by the following

Theorem II.

If there be a series, a . sine of $pz-b$. sine of $\overline{p+q}.z+c$. sine of $\overline{p+2q}.z-d$. sine of $\overline{p+3q}.z$ &c.$=s$, then shall

a . sine of $p'z-a'$. sine of $\overline{p'+q}.z+b'$. sine of $\overline{p'+2q}.z$ &c.$=s'$

a sine of $p''z-a''$ sine of $\overline{p''+q}.z+b''$ sine of $\overline{p''+2q}.z$ &c.$=s''$

a sine of $p^{n}z-a^{n}$ sine of $\overline{p^{n}+q}.z+b^{n}$ sine of $\overline{p^{n}+2q}.z$ &c.$=s^{n}$

&c. &c. &c. &c.

And if the series be,

a cos. of $px-b$ cos. of $\overline{p+q}\,.\,z+c$ cos. of $\overline{p+2q}\,.\,z$ &c. $=s$,

then shall,

a cos. of $p'z-a'$ cos. of $\overline{p'+q}\,.\,z+b'$ cos. of $\overline{p'+2q}\,.\,z$ &c. $=s'$

a cos. of $p''z-a''$ cos. of $\overline{p''+q}\,.\,z+b''$ cos. of $\overline{p''+2q}\,.\,z$ &c. $=s''$

&c. &c. &c. &c.

a', a'', a''', &c. b', b'', b''', &c. c', c'', c''', &c. &c. being formed from a, b, c, d, e, &c. as in *Theorem* I. p', p'', p''', &c. likewise as in *Theorem* I. $s'=2s$. cos. of $.\frac{1}{2}qz$, $s''=2s'$. cos. of $.\frac{1}{2}qz$, $s'''=2s''$. cos. of $\frac{1}{2}qz$, &c.

First, if a sine of $pz-b$ sine of $\overline{p+q}\,.\,z+c$ sine of $\overline{p+q}\,.\,z$ &c. $=s$, by multiplying by 2 cos. of $\frac{1}{2}qz$, by lemma No. II. we shall have a. sine of $\overline{p-\frac{1}{2}q}\,.\,z+a$. sine of $\overline{p+\frac{1}{2}q}\,.\,z-b$. sine of $\overline{p+\frac{1}{2}q}.z-b$. sine of $\overline{p+\frac{1}{2}q}.z+c$. sine of $\overline{p+\frac{1}{2}q}\,.\,z$ &c. $=s$. 2 cos. of $\frac{1}{2}qz$; consequently, putting $b-a=a'$, $c-b=b'$, &c. $p-\frac{1}{2}q=p'$, $s'=2s$ cos. of $\frac{1}{2}qz$, we have, a sine of $p'z-a'$ sine of $\overline{p'+q}\,.\,z+b'$ sine of $\overline{p'+2q}\,.\,z$ &c. $=s'$, which being exactly similar in form to the original series, the other series will be deduced from this by continually proceeding in the same method.

Again, if a . cos. of $pz-b$. cos. of $\overline{p+q}\,.\,z+c$. cos. of $\overline{p+2q}\,.\,z$ &c. $=s$, we have by multiplying by 2 cos. of $\frac{1}{2}qz$ by the help of lemma No. III., a cos. of $p'z-a'$ cos. of $\overline{p'+q}\,.\,z+b'$ cos. of $\overline{p'+2q}\,.\,z$ &c. $=s$, which being exactly similar in form, to the original, we may obtain the other series, which are likewise similar in form by the same mode of proceeding.

Cor. the πth successive value of $s=s.2$ cos. of $\frac{1}{2}qz|^{\pi}$, the πth successive value of $p=p-\pi.\frac{1}{2}$ and $s=\frac{\text{$\pi$th successive value of } s}{2 \text{ cos. of } \frac{1}{2}qz|^{\pi}}$

Example 1, Required the sum of the series, sine of $pz-$

sine of $\overline{p+q}.z+$ sine of $\overline{p+2q}z-$ &c. and likewise of, cos. of $pz-$ cos. of $\overline{p+q}.z+$ cos. of $\overline{p+2q}.z-$ &c.

Here in both, $a,\ b,\ c,\ d,$ &c. $=1,\ 1,\ 1,\ 1,$ &c. } therefore, in
$a',\ b',\ c',$ &c. $=1,\ 0,\ 0,\ 0,$ &c. } the first series we have, s' or $s\ .\ 2$ cos. of $\frac{1}{2}qz =$ sine of $p'z$ and $\therefore s = \frac{\text{sine of } p'z}{2 \text{ cos. of } \frac{1}{2}qz} = \frac{\text{sine of } \overline{p-\frac{1}{2}q}z}{2 \text{ cos. of } \frac{1}{2}qz}$, and for the second series we have, s', or, $s\ .\ 2$ cos. of $\frac{1}{2}qz =$ cos. of $\overline{p-\frac{1}{2}q}\ .\ z$, and therefore, $s = \frac{\text{cos. of } \overline{p-\frac{1}{2}q}.z}{2 \text{ cos. of } \frac{1}{2}qz}$.

Cor. 1. If $p=q$ the first series will be, sine of $pz-$ sine of $2pz+$ sine of $3pz$ &c. $= \frac{\text{sine of } \frac{1}{2}pz}{2 \text{ cos. of } \frac{1}{2}pz} = \frac{1}{2}$ tangent of $\frac{1}{2}pz$, and the second, cos. of $pz -$ cos. of $2pz +$ cos. of $3pz$ &c. $= \frac{\text{cos. of } -\frac{1}{2}pz}{2 \text{ cos. of } \frac{1}{2}pz} = \frac{1}{2}$.

Scholium. Though we have given two theorems, the one for a series whose terms are all positive, and the other for a series whose terms are alternately positive and negative; they are both true whatever the signs of the terms be, provided that proper signs be used in the operation; that is, if any term should have a contrary sign, to the sign of that term contained in the enunciation of the theorem used, then a contrary sign must likewise be prefixed to it in the operation; thus, for instance, if for a series whose terms are all positive we should use *Theorem* II. or for a series whose terms are alternately positive and negative we should use *Theorem* I., for $a,\ b,\ c,\ d,$ &c. we must write $a, -b, c, -d,$ &c. and therefore

$a,\ a',\ b',$ &c. $= a, -a+b, \overline{c+b}, \overline{-d+c},$ &c. $=$ (suppose) $a, -\alpha, \beta, -\gamma,$ &c.
$a,\ a',\ b',$ &c. $= a, -\alpha+a, \beta+a, -\gamma+a, \delta+a,$ &c. $= a, a_1, b_1, c_1, d_1,$ &c.
$a,\ a'',\ b'',$ &c. $= a, -a_1+a, b_1+a, -c_1+b_1,$ &c. $=$ - - - - - - &c.
&c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

and consequently we shall get new series, the coefficients of whose terms are formed from the coefficients in the preceding series, by addition instead of subtraction; and may be of good purpose on some occasions. And if we alternately use these theorems the operation will be performed by alternately taking sums and differences; and this will amount to the same as taking the differences of the alternate terms, beginning always with two noughts: but, for the more readily comprehending this, we shall offer a theorem which moreover is the first of these theorems I discovered, but previously thereto shall propose

Example 2. Let the series be either of these, sine of $px+r$ sine of $\overline{p+q}.x+r.\frac{r+1}{2}$ sine of $\overline{p+2q}.x+r.\frac{r+1}{2}.\frac{r+2}{3}$ sine of $\overline{p+3q}.x+$ &c., cos. of $px+r$ cos. of $\overline{p+q}.x+r.\frac{r+1}{2}$, cos. of $\overline{p+2q}.x+$ &c., sine of $px-r$ sine of $\overline{p+q}.x+r.\frac{r+1}{2}$ sine of $\overline{p+2q}.x-$ &c. or, cos. of $px-r$ cos. of $\overline{p+q}.x+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.x-$ &c. r being a whole positive number, the terms in the two first series all positive, and in the two last alternately positive and negative.

The coefficients being,

Of 1st term. 2d. 3d. 4th. 5th.

$$1,\ r,\ r.\frac{r+1}{2},\ r.\frac{r+1}{2}.\frac{r+2}{3},\ r.\frac{r+1}{2}.\frac{r+2}{3}.\frac{r+3}{4},\ \&c.$$

the first differences

$$1,\ \frac{r-1}{1},\ \frac{r-1}{1}.\frac{r}{2},\ \frac{r-1}{1}.\frac{r}{2}.\frac{r+1}{3},\ \frac{r-1}{1}.\frac{r}{2}.\frac{r+1}{3}.\frac{r+2}{4},\ \&c.$$

2d differences

$$1,\ \frac{r-2}{1},\ \frac{r-2}{1}.\frac{r-1}{2},\ \frac{r-2}{1}.\frac{r-1}{2}.\frac{r}{3},\ \frac{r-2}{1}.\frac{r-1}{2}.\frac{r}{3}.\frac{r+1}{4},\ \&c.$$

9d differences

Of 1st term. 2d. 3d. 4th. 5th.

$$1, \frac{r-1}{1}, \frac{r-1}{1} \cdot \frac{r-2}{2}, \frac{r-1}{1} \cdot \frac{r-2}{2} \cdot \frac{r-3}{3}, \frac{r-1}{1} \cdot \frac{r-2}{2} \cdot \frac{r-3}{3} \cdot \frac{r-4}{4}, \text{ &c.}$$

And in general, nth differences

$$1, \frac{r-n}{1}, \frac{r-n}{1} \cdot \frac{r-n-1}{2}, \frac{r-n}{1} \cdot \frac{r-n-1}{2} \cdot \frac{r-n-2}{3}, \text{ &c.}$$

to be continued to $n+1$ terms, and the remaining terms will be the $n+1$th term multiplied by $\frac{r}{n+1}, \frac{r}{n+1} \cdot \frac{r+1}{n+2}, \frac{r}{n+1} \cdot \frac{r+1}{n+2} \cdot \frac{r+2}{n+3}$, &c. and consequently if n be $= r$, all the terms of the nth differences except the first will vanish. Hence we have by *Theorem* I. and its *Cor.* 1. the sum of the series, sine of $pz+r$ sine of $\overline{p+q} \cdot z+r \cdot \frac{r+1}{2}$ sine of $\overline{p+2q} \cdot z+$ &c. $= \pm \frac{\text{sine of } \overline{p-\frac{1}{2}rq} \cdot z}{2 \text{ sine of } \frac{1}{2}qz}$, if r be even, but $\pm \frac{\text{cos. of } \overline{p-\frac{1}{2}rq} \cdot z}{2 \text{ sine of } \frac{1}{2}qz}$, if r be odd, the upper signs to be taken when r being divided by 4 leaves 0 or 1, and the under signs when it leaves 2 or 3. And the sum of the series, cos. of $pz+r$ cos. of $\overline{p+q} \cdot z+r \cdot \frac{r+1}{2}$ cos. of $\overline{p+2q} \cdot z+$, &c. $= \pm \frac{\text{sine of } \overline{p-\frac{1}{2}rq} \cdot z}{2 \text{ sine of } \frac{1}{2}qz}$ if r be odd, but $\pm \frac{\text{cos. of } \overline{p-\frac{1}{2}rq} \cdot z}{2 \text{ sine of } \frac{1}{2}qz}$ if even, the upper signs to be taken, if r leaves 3 or 0 when divided by 4, and the under if it should leave 2 or 1. In deducing the sum of this series from the said *Cor.* it is necessary to put $\overline{p+\frac{1}{2}q}$ for p and $r+1$ for the n used there. The sum of the series, sine of $pz-r$ sine of $\overline{p+q} \cdot z+r \cdot \frac{r+1}{2}$ sine of $\overline{p+2q} \cdot z-$, &c. by *Theorem* II. is $= \frac{\text{sine of } \overline{p-\frac{1}{2}rq} \cdot z}{2 \text{ cos. of } \frac{1}{2}qz}$; and the sum of the series, cos. of $pz-r$ cos. of $\overline{p+q} \cdot z+r \cdot \frac{r+1}{2}$ cos. of $\overline{p+2q} \cdot z-$ &c. by the same $= \frac{\text{cos. of } \overline{p-\frac{1}{2}rq} \cdot z}{2 \text{ cos. of } \frac{1}{2}qz}$.

Corollary. Because sine of $pz = pz - \frac{p^3z^3}{2.3} + \frac{p^5z^5}{2.3.4.5}$, &c. sine of $\overline{p+q}z = \overline{p+q}.z - \frac{\overline{p+q}^3}{2.3}.z^3 + \frac{\overline{p+q}^5}{2.3.4.5}z^5$, &c. sine of $\overline{p+2q}.z = \overline{p+2q}.z - \frac{\overline{p+2q}^3}{2.3}.z^3 + \frac{\overline{p+2q}^5}{2.3.4.5}.z^5$, &c. &c., and $\frac{1}{\text{cos. of } \frac{1}{2}qz} = \frac{1}{1-\frac{1}{2}q^2z^2 + \frac{1}{1.2.3.4}q^4z^4 \text{ &c.}} = 1 + Az^2 + Bz^4 + Cz^6$, &c.

Where A, B, C, &c. stand for the coefficients of the multinomial, $1 - \frac{1}{2}q^2z^2 + \frac{1}{1.2.3.4}q^4z^4$ &c. raised to the $-r$ power, and consequently r only concerned in them by pure powers; hence this being multiplied by $\overline{p-\frac{1}{2}qr}.z - \frac{\overline{p-\frac{1}{2}qr}^3.z^3}{2.3} + \frac{\overline{p-\frac{1}{2}qr}^5.z^5}{2.3.4.5}$ &c. the value of sine of $\overline{p-\frac{1}{2}qr}.z$, we obtain from the equation sine of $pz - r$. sine of $\overline{p+q}.z + r.\frac{r+1}{2}$ sine of $\overline{p+2q}.z$, &c. $= \frac{\text{sine of } \overline{p-\frac{1}{2}qr}.z}{2 \text{ cos. of } \frac{1}{2}qz}$, the series $\overline{p-r}.\overline{p+q+r}.\frac{r+1}{2}.\overline{p+2q}$ &c. $.z - \overline{p'-r}.\overline{p+q}^3 + r.\frac{r+1}{2}\overline{p+2q}^5$ &c. $.\frac{z^3}{2.3} + \overline{p'-r}.\overline{p+q}^3 + r\frac{r+1}{2}.\overline{p+2q}^5$ &c. $.\frac{z^5}{2.3.4.5}$, &c. $= \frac{p-\frac{1}{2}qr}{z'}z - \frac{\overline{p-\frac{1}{2}qr}^3}{z'} - \frac{2.3A}{z'}.\overline{p-\frac{1}{2}qr}$ $.\frac{z^3}{2.3} + \frac{\overline{p-\frac{1}{2}q}^5}{z'} - \frac{4.5A}{z'}.\overline{p-\frac{1}{2}qr}^3 + \frac{B}{z'}.\overline{p-\frac{1}{2}qr}.\frac{z^5}{2.3.4.5}$ &c., the law of continuation being evident in both series, consequently by comparing the homologous terms we obtain the sum of the series, $p-r.\overline{p+q}+r.\frac{r+1}{2}.\overline{p+2q}-r.\frac{r+1}{2}\frac{r+2}{3}.\overline{p+3q}$ &c. $= \frac{p-\frac{1}{2}qr}{z'}$, of $p'-r.\overline{p+q}^3 + r.\frac{r+1}{2}.\overline{p+2q}^5$ &c. $= \frac{\overline{p-\frac{1}{2}qr}^3}{z'} - \frac{2.3}{z'}.\overline{p-\frac{1}{2}qr}$, of the series $p'-r.\overline{p+q}^3 + r.\frac{r+1}{2}.\overline{p+2q}^5$ — &c. $= \frac{\overline{p-\frac{1}{2}q}^5}{z'} - \frac{4.5.A}{2}.\overline{p-\frac{1}{2}qr}^3 + \frac{2.3.4.5.B}{z'}.\overline{p-\frac{1}{2}qr}$, and so for the other odd powers, r being only concerned in these expressions

by pure powers, and by similar means we may from the equation, cos. of $pz-r$.cos. of $\overline{p+q}\,.\,z+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}\,.\,z$ &c. $=\frac{\text{cos. of }\overline{p-\frac{1}{2}q}\,rz}{2\text{ cos. of }\frac{1}{2}qz}$, obtain the series $1-r+r.\frac{r+1}{2}-r.\frac{r+1}{2}.\frac{r+2}{3}$ &c. $-p^r-r.\overline{p+q}\,^r+r.\frac{r+1}{2}.\overline{p+2q}\,^r-$&c. $\frac{z^2}{1.2}+p^r-r.\overline{p+q}\,^r+r.\frac{r+1}{2}.\overline{p+2q}\,^r-$&c. $\frac{z^4}{1.2.3.4}-$&c. $=\frac{1}{2^r}-\frac{\overline{p-\frac{1}{2}q}\,^r-1.2.A}{2^r}.\frac{z^2}{1.2}+\frac{\overline{p-\frac{1}{2}q}\,^r-3.4.A.\overline{p-\frac{1}{2}q}\,^r+1.2.3.4.B}{2^r}.\frac{z^4}{1.2.3.4}$; and consequently by again comparing the homologous terms, we find $1-r+r.\frac{r+1}{2}-r.\frac{r+1}{2}.\frac{r+2}{3}$ &c. $=\frac{1}{2^r}$, as it is well known to be, $p^r-r.\overline{p+q}\,^r+r.\frac{r+1}{2}\overline{p+2q}\,^r-$ &c. $=\frac{\overline{p-\frac{1}{2}q}\,^r-1.2.A}{2^r}$, $p^r-r.\overline{p+q}\,^r+r.\frac{r+1}{2}\overline{p+2q}\,^r$ &c. $=\frac{\overline{p-\frac{1}{2}q}\,^r-3.4.A.\overline{p-\frac{1}{2}q}\,^r+1.2.3.4.B}{2^r}$, and so for the other even powers, r being only concerned in these expressions by pure powers.

Hence r being a whole positive number, the sum of the series, $p^m-r.\overline{p+q}\,^m+r.\frac{r+1}{2}.\overline{p+2q}\,^m-r.\frac{r+1}{2}.\frac{r+2}{3}.\overline{p+3q}\,^m$ &c, m likewise being a whole positive number, may be always expressed by $\frac{1}{2^r}\times$ by a series of finite terms of pure powers of r whose coefficients are given, of the form $a+br+cr^2$ &c. p, q, and m being given values, and a, b, c, &c. determinate values independent of r; merely by comparing the coefficients of the homologous powers, of z, in the two equations of the series above. Now if we can prove that the same expressions, derived from the comparison of the coefficients of the homologous powers of z, give the sum of the series $p^m-r.\overline{p+q}\,^m+r.\frac{r+1}{2}.\overline{p+2q}\,^m$ &c. whether r be a whole positive number

or not, it will follow that the series, sine of $px-r$ sine of $\overline{p+q}.x+r.\frac{r+1}{2}$ sine of $\overline{p+2q}.x$ &c. will be equal to $\frac{\text{sine of } p-\frac{1}{2}qr.x}{2\text{ cos. of }\frac{1}{2}qx|^r}$ and the series, cos. of $px-r$ cos. of $\overline{p+q}.x+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.x$ &c. $=\frac{\text{cos. of } p-\frac{1}{2}qr.x}{2\text{ cos. of }\frac{1}{2}qx|^r}$, whether r be a whole positive number or not.

And in order to prove this requisite, we shall first premise that if we have the sum of the series $p^n-r.\overline{p+q}|^n+r.\frac{r+1}{2}.\overline{p+2q}|^n-r.\frac{r+1}{2}.\frac{r+2}{3}.\overline{p+3q}|^n$ &c. whatever r may be, (m, p and q being given quantities) expressed by a series, $\frac{1}{2^r}\times\overline{A+Br+Cr^2}$ &c. of finite terms in which the *functional* values of p and r are distinct, A, B, C, &c. being given quantities independent of r, we may likewise find the sum of the series $p^{m+1}-r.\overline{p+q}|^{m+1}+r.\frac{r+1}{2}\overline{p+2q}|^{m+1}$ &c. for this series is equal to $p.p^n-r.\overline{p+q}.\overline{p+q}|^n+r.\frac{r+1}{2}.\overline{p+2q}.\overline{p+2q}|^n$ &c. $=p\times\overline{p^n-r.\overline{p+q}|^n+r.\frac{r+1}{2}.\overline{p+2q}|^n}$ &c. $-rq\times\overline{\overline{p+q}|^n-r+1.\overline{p+2q}|^n+r+1.\frac{r+2}{2}.\overline{p+3q}|^n-r+1.\frac{r+2}{2}.\frac{r+3}{3}.\overline{p+4q}|^n}$ &c. but $p^n-r.\overline{p+q}|^n+r.\frac{r+1}{2}.\overline{p+2q}|^n$ &c. is equal to $\frac{1}{2^r}\times\overline{A+Br+Cr^2}$ &c. and if in this we write $r+1$ for r and $\overline{p+q}$ for p, we shall have the sum of the series $\overline{p+q}|^n-r+1.\overline{p+2q}|^n+r+1.\frac{r+2}{2}.\overline{p+3q}|^n$ &c. $=\frac{1}{2^{r+1}}\times A,+B,.\overline{r+1}+C,.\overline{r+1}|^2$ &c., A, B, C, &c. standing for the values that A, B, C, &c. become by writing $\overline{p+q}$ for p; and this may evidently be reduced to an expression of the finite terms of the form $\frac{1}{2^r}\times\overline{A'+B'r+C'r^2}$ &c. and con-

sequently will the sum of the series $p^{m+1}-r.\overline{p+q}^{m+1}+r.\frac{r+1}{2}.\overline{p+2q}^{m+1}$ &c. be = the expression $\frac{1}{2^r}\times\overline{pA+pBr+pCr^2}$ &c. $-qA'r-qB'r^2$ &c. of finite terms. This being proved, it follows, because the sum of the series $p^m-r.\overline{p+q}^m+r.\frac{r+1}{2}.\overline{p+2q}^m$ &c. when m is equal to 0, is equal to $1-r+r.\frac{r+1}{2}$ &c. =, by the binomial theorem, $\frac{1}{2^r}$, whatever r may be, that the sum of the series $p-r.\overline{p+q}+r.\frac{r+1}{2}\overline{p+2q}$ &c. namely, the said series when m is equal to 1, may be expressed by $\frac{1}{2^r}\times$ a series of pure powers of r of a finite number of terms whatever r may be, and comes out by the bye $\frac{1}{2^r}\times\overline{p-\frac{1}{2}qr}$, the same as above, and consequently by writing 1, 2, 3, 4, 5, 6, &c. one after the other for, m, we shall find that the sum of the series $p^m-r.\overline{p+q}^m+r.\frac{r+1}{2}\overline{p+2q}^m$ &c. may always be expressed by $\frac{1}{2^r}$ multiplied by a series of finite terms in the form $A+Br+Cr^2$ &c. A, B, C, &c. p, q, m, &c. being independent of r; and m a whole positive number. And these will, we shall prove without running through all the infinite cases, be the very same expressions as those given above, by comparing the coefficients of the homologous powers of x. In order to this we observe, since we have just proved that the sum of the said series, whatever r may be, may be expressed by $\frac{1}{2^r}\times$ series $\overline{A+Br+Cr^2}$ &c. of a finite number of terms, and from the comparison of the homologous powers, that when r is a whole number it may be expressed by $\frac{1}{2^r}\times$ value $\overline{a+br+cr^2}$ &c. of a finite number of terms, it follows that when r is any

whole number, that these two values must then be equal to each other, and ∴ that $A+Br^{s}+Cr^{s}$ &c. containing a finite number of terms must be then equal to $a+br+cr^{s}$ &c. containing a finite number of terms, and consequently the highest power of r and its coefficient must be the same in both series, otherwise by increasing r by the same in both, one side of the equation would become greater than the other, which is absurd; consequently the highest power of r and its coefficient is the same in both, and will destroy each other, and consequently the next highest powers of r and likewise their coefficients must be the same with each other, and will therefore be destroyed, &c. Hence the powers of r and their respective coefficients being the same in both, the expressions themselves must be the same in every respect, whether r be a whole number or not.

Hence we have not only given two different means of summing the series $p^{n}-r\,.\,\overline{p+q}|^{m}$ &c. (m being a whole positive number) whatever r may be, which indeed was not our chief object, but we have likewise proved that the series sine of $pz-r$ sine of $\overline{p+q}.z$ &c. $=\frac{\text{sine of }\overline{p-\frac{1}{2}q}.r.z}{2\text{ cos. of }\frac{1}{2}qz|^{r}}$, and the series cos. of $pz-r$ cos. of $\overline{p+q}.z$ &c. $=\frac{\text{cos. of }\overline{p-\frac{1}{2}q}.r.z}{2\text{ cos. of }\frac{1}{2}qz|^{r}}$, whatever r may be, the same as LANDEN finds.

Cor. II. Because these two series are equally true, whatever p may be, if for p we write $qr-p$ throughout, in the first we shall have, sine of $\overline{qr-p}\,.\,z-r$ sine of $\overline{r+1}\,.\,q-p\,.\,z+r\,.\,\frac{r+1}{2}$ sine of $\overline{r+2}.q-p.z$ &c. $=\frac{\text{sine of }\frac{1}{2}qr-p.z}{2\text{ cos. of }\frac{1}{2}qz|^{r}}=-\frac{\text{sine of }\overline{p-\frac{1}{2}q}\,r.z}{2\text{cos. of }\frac{1}{2}q|^{r}}$.

Consequently sine of $\overline{qr-p}.z-r$ sine of $\overline{r+1}\,q-p.z+$ &c.

$= -$ sine of $\overline{pz+r}$ sine of $\overline{p+q}.z-$, &c. and from cos. of $pz-r$ cos. of $\overline{p+q}.z$, &c. $= \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.z}{2 \text{ cos. of } \frac{1}{2}qz}$ by the like substitution we get cos. of $\overline{qr-p}.z-r$ cos. of $\overline{r+1}.q-p.z+r.\frac{r+1}{2}$ cos. of $\overline{r+2}.q-p.z$ &c. $= \left(\frac{\text{cos. of } \overline{\frac{1}{2}qr-p}.z}{2 \text{ cos. of } \frac{1}{2}qz} = \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.z}{2 \text{ cos. of } \frac{1}{2}qz}=\right)$ cos. of $pz-r$ cos. of $\overline{p+q}.z+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.z$ &c. and these by *Cor.* I. are true, whatever r may be.

Cor. III. If p be $=\frac{1}{2}qr$ we shall have, cos. of $\frac{1}{2}qrz-r$ cos. of $\frac{1}{2}q.\overline{r+2}.z+r.\frac{r+1}{2}.$ cos. of $\frac{1}{2}q.\overline{r+4}.z$ &c. $= \frac{\text{cos. of } o}{2 \text{ cos. of } \frac{1}{2}qz} = \frac{1}{2 \text{ cos. of } \frac{1}{2}qz}$, or if A be written for $\frac{1}{2}qz$ we shall have, cos. of $r\text{A}-r$ cos. of $\overline{r+2}\,\text{A}+r.\frac{r+1}{2}$ cos. of $\overline{r+4}.\text{A}-$ &c. $= \frac{1}{2 \text{ cos. of A}}$ which is the same in substance as SIMPSON's lemma, page 67 of his Tracts.

Cor. IV. If we put $pz = 180^\circ - tz$, $qz = 180^\circ - sz$ we shall have according to *Scholium* III. at the end of the examples to *Theorem* I.

sine of $tz+r$ sine of $\overline{t+s}.z+r.\frac{r+1}{2}$ sine of $\overline{t+2s}.z$ &c. $= \frac{\text{sine of } \overline{180^\circ-tz-\frac{1}{2}r}.\overline{180^\circ-sz}}{2 \text{ cos. of } 90^\circ-\frac{1}{2}qz} = \frac{\text{sine of } \overline{90^\circ.r+t-\frac{1}{2}rs}.z}{2 \text{ sine of } \frac{1}{2}sz}$ and $-$ cos. of $tz-r$ cos. of $\overline{t+s}.z-r.\frac{r+1}{2}$ cos. of $\overline{t+2s}.z$ &c. $= \frac{\text{cos. of } \overline{180^\circ-tz-\frac{1}{2}r}.\overline{180^\circ-sz}}{2 \text{ cos. of } 90^\circ-\frac{1}{2}sz} = -\frac{\text{cos. of } \overline{90^\circ+t-\frac{1}{2}rs}.z}{2 \text{ of sine } \frac{1}{2}sz}$ $\therefore$ cos. of $tz+r$ cos. of $\overline{t+s}.z+r.\frac{r+1}{2}$ cos. of $\overline{t+2s}.z$ &c. $= \frac{\text{cos. of } \overline{90^\circ r+t-\frac{1}{2}rs}.z}{2 \text{ sine of } \frac{1}{2}sz}$

and if in these r be a whole number, and p and q be written

for t and s, we shall have the results determined above. Many more corollaries may be derived from these.

Theorem III.

If there be formed a series of terms

$a, b, c, d, e, f,$ &c.
$a, b, a', b', c', d',$ &c.
$a, b, a'', b'', c'', d'',$ &c.
$a, b, a''', b''', c''', d''',$ &c.
&c, &c, &c, &c, &c, &c, &c.

The terms of each series being formed from those immediately above, by taking the alternate differences of the terms, always beginning with 0, 0; that is, taking 0 from the 1st term, 0 from the 2d term, 1st term from 3d term, 2d term from 4th term, &c. in any of the series, for 1st, 2d, 3d, &c. terms of the next series. And p' be put $= p - q$, $p'' = p' - q$, $p''' = p'' - q$, &c. $s' = s \cdot 2 \sin. of qz$, $s'' = -s' \cdot 2 \sin. of qz$, $s''' = s'' \cdot 2 \sin. of qz$, $s^{iv} = -s''' \cdot 2 \sin. of qz$, &c. I say if there be a series $a \sin. of pz + b \sin. of \overline{p+q} \cdot z + c \sin. of \overline{p+2q} \cdot z + d \sin. of \overline{p+3q} \cdot z$, &c. $= s$, we shall have, $a \cos. of p'z + b \cos. of \overline{p'+q} \cdot z + a' \cos. of \overline{p'+2q} \cdot z + b' \cos. of \overline{p+3q} \cdot z$, &c. $= s'$ $a \sin. of p''z + b \sin. of \overline{p''+q} \cdot z + a'' \sin. of \overline{p''+2q} \cdot z + b'' \sin. of \overline{p+3q} \cdot z$, &c. $= s''$, &c.

For multiplying the first of these by $2 \sin. of qz$, by help of lemma No. I. we shall have, $a \cdot \cos. of \overline{p-q} \cdot z - a \cdot \cos. of \overline{p+q} \cdot z + b \cdot \cos. of pz - b \cdot \cos. of \overline{p+2q} \cdot z + c \cos. of \overline{p+q} \cdot z - c \cos. of \overline{p+3q} \cdot z + d \cos. of \overline{p+2q} \cdot z - d \cos. of \overline{p+4q}$

$. z + r$ cos. of $\overline{p + 3q} . z$, &c. $= s . 2$ sine of qz, therefore putting $a' = c - a$, $b' = d - b$, $c' = e - c$, &c. $p' = p - q$, $s' = s . 2$ sine of qz, we get a cos. of $p'z + b$ cos. of $\overline{p' + q} . z + a'$ cos. of $\overline{p' + 2q} . z + b'$ cos. of $\overline{p + 3q} . z$, &c. $= s'$, and multiplying this by 2 sine of qz, by help of lemma No. II. we get $a .$ sine of $\overline{p' + q} z - a$ sine of $\overline{p' - q} . z + b$ sine of $\overline{p' + 2q} . z - b$ sine of $p'z + a'$ sine of $\overline{p' + 3q} . z - a'$ sine of $\overline{p' + q} . z + b'$ sine of $p' + 4q . z - b'$. sine of $\overline{p' + 2q} . z$ &c. $= s' . 2$ sine of qz, therefore putting $a' - a = a''$, $b' - b = b''$, $c' - a' = c''$ &c. $p' - q = p''$, $- s' . 2$ sine of $qz = s''$ we have, $a .$ sine of $p''z + b .$ sine of $\overline{p'' + q} . z + a''$. sine of $\overline{p'' + 2q} . z + b''$. sine of $\overline{p'' + 3q} . z$ &c. $= s''$, which being exactly similar in form to the original series, the successive series, which will be of a similar form to the second or first of the series, will be deduced by the like operations and substitutions. Q. E. D.

Corollary 1. The nth successive value of s is $= \pm s . \overline{2 \text{ sine of } qz}|^n$ or $\pm s' . \overline{2 \text{ sine of } qz}|^{n-1}$, the upper sign to be taken when n being divided by 4 leaves 0 or 1, otherwise the under sign and the nth successive value of $p = p - n . q$.

Corollary 11. These operations are performed by differences whether the signs be all positive, or alternately positive and negative.

Example 1. Required the sum of the series n sine of $pz + \overline{n + r}$ sine of $\overline{p + q} . z + \overline{n + 2r}$ sine of $\overline{p + 2q} . z$ &c.

Here $a, b, c, d,$ &c. $= n, n + r, n + 2r, n + 3r, n + 4r,$ &c.
$a, b, a', b',$ &c. $= n, n + r, 2r, 2r, 2r,$ &c.
$a, b, a'', b'',$ &c. $= n, n + r, 2r - n, r - n, 0,$ &c.

$\therefore s'' = - s . \overline{2 \text{ sine of } qz}|^n = n$ sine of $p''z + \overline{n + r}$ sine of $\overline{p'' + q}$

$z + \overline{2r-n}$ sine of $p'' + 2q \,.\, z + r - n$ sine of $p'' + 3q \,.\, z$ &c. by restoration and division we have s the sum $= [n$ sine of $p - 2q \,.\, z + \overline{n+r}$ sine of $p - q \,.\, z + \overline{2r-n}$ sine of $pz + \overline{r-n}$ sine of $p + q \,.\, z] \div -2$ sine of $qz|$: had we used *Theorem* I. we should have gotten a more simple valuation; namely, $s = \frac{n \text{ sine of } \overline{p-q} \,.\, z + \overline{r-n} \text{ sine of } pz}{-2 \text{ sine of } \frac{1}{2}qz|}$ which is reducible to the other by multiplying the upper and under terms by 2 cos. of $\frac{1}{2}qz|$ by help of lemma No. II. and III. Had the terms been alternate positive and negative we should have had

$$\left.\begin{array}{l} a, b, c, d, e, \text{ &c.} = n, -n+r, \quad n+2r, -n+3r, \quad n+4r, \text{ &c.} \\ a, b, a', b', c', \text{ &c.} = n, -n+r, + 2r, \quad - 2r, \quad + 2r, \text{ &c.} \\ a, b, a'', b'', c'', \text{ &c.} = n, -n+r, \quad 2r-n, -r-n, \quad 0, \quad \text{ &c.} \end{array}\right\}$$

and therefore $s = [n$ sine of $p - 2q \,.\, z, -n+r$ sine of $p - q \,.\, z + \overline{2r-n}$ sine of $pz - \overline{r-n}$ sine of $p+q \,.\, z] \div -2$ sine of $qz|$. If we had used *Theorem* II. we should have obtained $s = \frac{n \text{ sine of } \overline{p-q} \,.\, z - \overline{r-n} \,.\, \text{sine of } pz}{2 \text{ cos. of } \frac{1}{2}qz|}$ which is reducible to the other by multiplying the upper and under terms by 2 sine of $\frac{1}{2}qz|$, by help of lemma No. I. and II.

Theorem IV.

If there be a series, a . sine of $pz+b$. sine of $\overline{p+q} \,.\, z - c$. sine of $p+2q \,.\, z - d$. sine of $p+3q \,.\, z +$ &c. $= s$ or a cos. of $pz + b$ cos. of $\overline{p+q} \,.\, z - c$ cos. of $p+2q \,.\, z - d$. cos. of $\overline{p+3q} \,.\, z +$ &c. $= s$ the signs of the terms changing alternately two by two; then in the first case

a sine of $px + b$ sine of $\overline{p+q}.x - a'$ sine of $\overline{p+2q}.x - b'$ sine of $\overline{p+3q}.x +$ &c. $= s'$

a sine of $p''x + b$ sine of $\overline{p''+q}.x - a''$ sine of $\overline{p''+2q}.x - b''$ sine of $\overline{p''+3q}.x +$ &c. $= s''$

&c. &c. &c. &c.

and in the second case

a cos. of $px + b$ cos. of $\overline{p+q}.x - a'$ cos. of $\overline{p+2q}.x - b'$ cos. of $\overline{p+3q}.x +$ &c. $= s'$

a . cos. of $p'x + b$ cos. of $\overline{p''+q}.x - a''$ cos. of $\overline{p''+2q}.x - b'$ cos. of $\overline{p''+3q}.x +$ &c. $= s''$

&c. &c. &c. &c.

where the terms a, b, a', b', c', &c. a, b, a'', b'', c'', &c. &c. are formed by taking the alternate differences, as in the last theorem; p', p'', p''', &c. likewise as in that theorem, $s' = s$. 2 cos. of qx, $s'' = 2$ cos. of qx $s' - s$, $s''' = 2$ cos. of qx $s'' - s'$ &c.

This is plain by multiplying the series continually by 2 cos. of qx by help of lemma No. II. for case 1, and lemma No. III. for case 3.

Example. Required the sum of the series, sine of $x +$ sine of $2x -$ sine of $3x -$ sine of $4x +$ &c.

Here $p = q = 1$ $\left.\begin{matrix} a, b, c, d, \text{ &c.} = 1, 1, 1, 1, 1, \text{ &c.} \\ a, b, a', b', \text{ &c.} = 1, 1, 0, 0, 0, \text{ &c.} \end{matrix}\right\}$ $\therefore s' = s$. 2 cos. of x

$= \text{sine of } 0x + \text{sine of } x$. $\therefore s = \frac{\text{sine of } x}{2 \text{ cos. of } x}$.

Scholium 1. As the two first theorems depend on the differences of the coefficients of the immediate terms or omitting none, the two last on the differences of the coefficients of the alternate terms or omitting one term; so we may give theorems for the differences of the coefficients of

the terms, omitting 2, 3, &c. terms; in fact, if r be a whole number, and the terms of the series be all positive, or any how positive and negative by sets, provided the same signs return in the same order after every set, consisting of r number of terms; by continually multiplying by 2 sine of $\frac{r}{2}qz$, we shall get new series by taking the differences of the coefficients of every term and the rth succeeding term beginning with r number of noughts; except indeed that the coefficients of the terms will sometimes have the order of signs interrupted, namely, when a greater value is to be subtracted from a less.

But if every set should have the same order to signs contrary to those in the set immediately preceding, and consequently every set omitting one set continually, have the same order of signs, then by continually multiplying by 2 cos. of $\frac{r}{2}qz$, we shall get new series by taking the differences of the coefficients of any term and the rth term from it.

Scholium II. We may by the methods above not only find the valuation of infinite series, but likewise of finite series.

Example 1. Required the sum of the r first terms of the series, cos. of $nz+$ cos. of $\overline{n+q}.z+$ cos. of $\overline{n+2q}.z$ &c.

The series *ad infinitum* may be written thus, cos. of $nz+$ cos. of $\overline{n+q}.z+$ cos. of $\overline{n+2q}.z$ - - - - $+$ cos. of $\overline{n+r-1}.qz$ $+$ cos. of $\overline{n+r}.z+$ cos. of $\overline{n+r+1}.q.z+$ &c. *ad infinitum*, = from which if we take cos. of $\overline{n+r}.z+$ cos. of $\overline{n+r+1}.q.z$ $+$ cos. of $\overline{n+r+2}.q.z$ &c. *ad infinitum*, we shall have the required sum; the first of these by *Example* 2, *Theorem* I. $= -\frac{\text{sine of } \overline{n-1}.\frac{qz}{2}}{2 \text{ sine of } \frac{qz}{2}}$, and the second by the same, by merely

writing $n+rq$ in the room of n, is equal to $-\frac{\text{sine of } n+r-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz}$, consequently the sum of the r first terms $=$ $\frac{\text{sine of } n+r-\frac{1}{2}q.z - \text{sine of } n-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz}$.

Cor. 1. If $n=q=1$, and rz the whole circumference of the circle, we shall have cos. of $z+$ cos. of $2z+$ cos. of $3z$ - - - - $+$ cos. of $rz=\frac{\text{sine of } 360°+n-\frac{1}{2}q.z - \text{sine of } n-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz}=0$, a theorem said to be used by LE GENDRE in his inscription of a polygon of 17 sides; and if we have $rqz=$ to the whole circumference, we likewise have in general cos. of $nz+$ cos. of $n+q.z$ - - - - $+$ cos. of $n+r-1.q.z=0$, and if $n=\frac{1}{2}q$, we have in general cos. of $nz+$ cos. of $3nz+$ cos. of $5nz+$ &c. - - - - - cos. of $2r-1.nz=\frac{\text{sine of } 2rnz}{2 \text{ sine of } nz}$.

Example 2, Required the sum of the series, cos. of $nz-$ cos. of $n+q.z+$ cos. of $n+2q.z$ - - - - - $\pm$ cos of $n+r-1.q.z$ the upper sign to be taken if r be odd, and the under sign if even.

The series is evidently the difference between the series cos. of $nz-$ cos. of $n+q.z+$ &c. *ad infinitum* and $\mp$ cos. of $n+rq.z \pm$ cos. of $n+r+1.q.z$ &c. *ad infinitum*, by proper substitution in *Example* 1, *Theorem* II. we have their respective sums $\frac{\text{cos. of } n-\frac{1}{2}q.z}{2 \text{ cos. of } \frac{1}{2}qz}$ and $+\frac{\text{cos. of } n+r-\frac{1}{2}q.z}{2 \text{ cos. of } \frac{1}{2}qz}$, and the difference $=$ $\frac{\text{cos. of } n-\frac{1}{2}q.z \pm \text{cos. of } n+r-\frac{1}{2}q.z}{\text{cos. of } \frac{1}{2}qz}$, the upper sign to be taken if r be odd and the under if even.

Example 3, Required the sum of the r first terms of the series,

$t^r \cos.\ \text{of}\ pz-\overline{t+v}^r \cos.\ \text{of}\ \overline{p+q}.z+\overline{t+2v}^r \cos.\ \text{of}\ \overline{p+2q}.z-\overline{t+3v}^r \cos.\ \text{of}\ \overline{p+3q}.z$ &c.

Using *Theorem* II. to find the sum *ad infinitum*, and expanding the coefficients, we have,

$a, b, c, d,$ &c. $= t^r, t^r+2tv+v^r, t^r+4tv+4v^r, t^r+6tv+9v^r,$
$t^r+8tv+16v^r,$ &c.

$a, a', b', c',$ &c. $= t^r, \quad 2tv+v^r, \quad 2tv+3v^r, \quad 2tv+5v^r,$
$2tv+7v^r,$ &c.

$a, a'', b'', c'',$ &c. $= t^r, -t^r+2tv+v^r, \quad 2v^r, \quad 2v^r,$
$2v^r,$ &c.

$a, a''', b''', c''',$ &c. $= t^r, -2t^r+2tv+v^r, t^r-2tv+v^r, \quad 0,$
$0,$ &c.

Therefore s the sum *ad infinitum* $= \left[t^r \cos.\ \text{of}\ \overline{p-\frac{1}{2}q}.z+\overline{2t^r-2tv-v^r}.\cos.\ \text{of}\ \overline{p-\frac{1}{2}q}.z+\overline{t-v}^r \cos.\ \text{of}\ \overline{p+\frac{1}{2}q}.z\right] \div \overline{2\cos.\ \text{of}\ \frac{1}{2}qz}^2$, but the sum of r first terms of the series is evidently equal to the sum *ad infinitum* $\pm$ the sum of the series $\overline{t+rv}^r \cos.\ \text{of}\ \overline{p+rq}.z-\overline{t+r+1.v}^r \cos.\ \text{of}\ \overline{p+r+1.q}.z+$ &c. *ad infinitum*, which is found from the last by writing $t+rv$ for t, and $p+rq$ for p, to be $\left[\overline{t+rv}^r \cos.\ \text{of}\ \overline{p+r-\frac{1}{2}q}.z+\overline{2t^2+2tr-1.2tv+2r^2-2r-1.v^2}.\cos.\ \text{of}\ \overline{p+r-1.q}.z+\overline{t+r-1.v}^r.\cos.\ \text{of}\ \overline{p+r+\frac{1}{2}.q}.z\right] \div 2\cos.\ \text{of}\ \frac{1}{2}qz^2$, which added to, or subtracted from, the value above, according as r is odd or even, gives the sum of r first terms of the original series.

Cor. If $z=0$, the cosine of any multiple of z will be equal to 1, therefore the sum of r first terms of $\overline{t^r-\overline{t+v}^r+\overline{t+2v}^r}$ &c. will be equal to $\frac{t^r+2t^r-2tv-v^r+\overline{t-v}^r}{8} \pm \frac{\overline{t+rv}^r+2t^r+2tr-1.2tv+2r^2-2r-1.v^2+\overline{t+r-1.v}^r}{8} = \frac{t^r-tv}{2} \pm$

$\frac{r^r+\sqrt{r^2-1}.tv+t^{r-r}v^r}{2}$ + or − to be taken according as r is odd or even, that is, $t^r+\sqrt{r^2-1}.tv+\frac{r^r-r}{2}v^r$ if r be odd, and $-rtv-\frac{r^r-r}{2}.v^r$ if even. And thus we might proceed to the discovery of an infinite variety of theorems relative to the sines and cosines contained between any two limits in a circle, and the consequent inferences, the method being capable of a very extensive application; but rather than tire the reader's patience with what he may effect himself from what has been already said, if there should otherwise have been any difficulty, I shall propose

Theorem V.

If we have the sum of the series, a sine of $pz+b$ sine of $p+q.z+c$ sine of $p+2q.z+d$ sine of $p+3q.z$ &c. expressed generally in terms of p, q, and z, we may find the sum of the series, a cos. of $pz+b$ cos. of $p+q.z+c$ cos. of $p+2q.z+d$ cos. of $p+3q.z$ &c. expressed generally in terms of p, q, and z, and the contrary.

For if we put $90^\circ+pz$ for pz in the series, and in the expression for its sum, we shall have instead of the sum of the series, a sine of $pz+b$ sine of $p+q.z+c$ sine of $p+2q.z$ &c., the sum of the series, a. sine of $90^\circ+pz+b$. sine of $90^\circ+p+q.z+c$ sine of $90^\circ+p+2q.z$ &c. or because the sine of an arc is equal to the sine of $180^\circ-$ that arc, we shall have the sum of the series a. sine of $90^\circ-pz+b$. sine of $90^\circ-p+q.z$ &c. or its equal, a. cos. of $pz+b$. cos. of $p+q.z+c$. cos. of $p+2q.z$ &c. which is the first part of the theorem; and by following the

steps backwards, and substituting $px-90°$ for px throughout we get the second part Q. E. D.

This theorem evidently supposes that the *functional* values of px are distinct in the general expression for the sum of the series, before the substitution takes place, which may not be the case if p has any particular value, or even if p, q, and x have any relation to each other.

Theorem VI.

Given the sum of the series, a sine of $vx+b$ sine of $\overline{v+u}.x+c$ sine of $\overline{v+2u}.x+d$ sine of $\overline{v+3u}.x+$ &c. and likewise of a cos. of $vx+b$ cos. of $\overline{v+u}.x+c$ cos. of $\overline{v+2u}.x+$ &c. expressed generally and distinctly in terms of x for any particular values of v and u, except $u=0$, v and u having the same value in both series; there will likewise be given the sum of the series, a sine of $px+b$ sine of $\overline{p+q}.x+c$ sine of $\overline{p+2q}.x$ &c. and likewise of, a cos. of $px+b$ cos. of $\overline{p+q}.x+c$ cos. of $\overline{p+2q}.x$ &c. generally and distinctly in terms of p, q, and x.

For, calling the first series A and the second B, and putting $x=\frac{qx}{u}$, we have by substitution,

a sine of $\frac{qv}{u}x+b$ sine of $\overline{\frac{qv}{u}+q}.x+c$ sine of $\overline{\frac{qv}{u}+2q}.x+d$ sine of $\overline{\frac{qv}{u}+3q}.x+$ &c. $=$ A, and

a cos. of $\frac{qv}{u}x+b$ cos. of $\overline{\frac{qv}{u}+q}.x+c$ cos. of $\overline{\frac{qv}{u}+2q}.x+d$ cos. of $\overline{\frac{qv}{u}+3q}.x+$ &c. $=$ B.

A and B being now expressed in general terms of q and x, and particular values; multiply the first by, 2 cos. of $\overline{p-\frac{qv}{u}}.x$

and the second by 2 sine of $\overline{p-\frac{q\pi}{n}}.x$ by means of lemma No. II. and we get, a sine of $px-a$ sine of $\overline{p-\frac{2q\pi}{n}}.x+b$ sine of $\overline{p+q}.x-b$ sine of $\overline{p-\frac{2q\pi}{n}-q}.x+c$ sine of $\overline{p+2q}.x-c$ sine of $\overline{p-\frac{2q\pi}{n}-2q}.x$ &c.$=2$A cos. of $\overline{p-\frac{q\pi}{n}}.x$, and a sine of $px+a$ sine of $\overline{p-\frac{2q\pi}{n}}.x+b$ sine of $\overline{p+q}.x+b$ sine of $\overline{p-\frac{2q\pi}{n}-q}.x+c$ sine of $\overline{p+2q}.x+c$ sine of $\overline{p-\frac{2q\pi}{n}-2q}.x$ &c.$=2$B sine of $\overline{p-\frac{q\pi}{n}}x$; consequently, adding these two together, we have by dividing by 2, a sine of $px+b$ sine of $\overline{p+q}.x+c$ sine of $\overline{p+2q}.x$ &c.$=$A cos. of $\overline{p-\frac{q\pi}{n}}.x+$B sine of $\overline{p-\frac{q\pi}{n}}.x$, expressed generally and distinctly in terms of p, q, and x, the equation will therefore remain if we put z in the place of x throughout, and therefore the sum of the series, a sine of $pz+b$ sine of $\overline{p+q}.z$ &c. is given expressed generally in terms p, q, and of z, which is the first part of the theorem.

Again, by multiplying the series, a sine of $\frac{q\pi}{n}x+b$ sine of $\overline{\frac{q\pi}{n}+q}.x$ &c.$=$A, by 2 sine of $\overline{p-\frac{q\pi}{n}}x$, by means of lemma No. I. and the series, a cos. of $\frac{q\pi}{n}x+$ cos. of $\overline{\frac{q\pi}{n}+q}.x$ &c.$=$B, by 2 cos. of $\overline{p-\frac{q\pi}{n}}x$ by means of lemma No. II. we shall have a cos. of $\overline{p-\frac{q\pi}{n}}.x-a$ cos. of $px+b$ cos. of $\overline{p-\frac{2q\pi}{n}-q}.x-b$ cos. of $\overline{p+q}.x+$ cos. of $\overline{p-\frac{2q\pi}{n}-2q}.x-c$ cos. of $\overline{p+2q}.x+$ &c.$=2$A cos. of $\overline{p-\frac{q\pi}{n}}x$, and a cos. of $\overline{p-\frac{q\pi}{n}}.x+a$ cos. of $px+b$ cos. of $\overline{q-\frac{2q\pi}{n}-q}.x+b$ cos. of $\overline{p+q}.x+c$ cos. of $\overline{q-\frac{2q\pi}{n}-2q}$

$.x+c$ cos. of $\overline{p+2q}.x+$ &c. $=aB$ cos. of $p-\frac{\pi}{2}.x$; half the difference of these two series gives,

a cos. of $px+b$ cos. of $\overline{p+q}.x+c$ cos. of $\overline{p+2q}.x+$ &c. $=B$ cos. of $p-\frac{\pi}{2}.x-A$ sine of $p-\frac{\pi}{2}.x$ expressed generally and distinctly in terms of p, q, and x; and consequently by writing z for x throughout, we have the sum of the series, a cos. of $pz+b$ cos. of $\overline{p+q}.z+c$ cos. of $\overline{p+2q}.z+$ &c. expressed generally and distinctly in terms z, p, and q. Q. E. D.

Cor. I. It is evident that p and q may be taken any numbers either positive or negative, but π ought not to be equal to 0, for we could not then effect the substitution $z=\frac{qx}{\pi}$.

Cor. II. Putting, a cos. of $pz+b$ cos. of $\overline{p+q}.z$ &c. $=P$, and a sine of $pz+b$ sine of $\overline{p+q}.z$ &c. $=Q$, and also B' and A' for the values that B and A become, by writing π for x in those values, that is, z for $\frac{\pi z}{q}$ in the given expressions B and A we shall have $P=B'$ cos. of $p-\frac{\pi}{2}.z-A'$ sine of $p-\frac{\pi}{2}.z$, and $Q=B'$ sine of $p-\frac{\pi}{2}.z+A'$ cos. of $p-\frac{\pi}{2}.z$.

Cor. III. Hence we may again prove, that if we have the sum of the series, a sine of $px+b$ sine of $\overline{p+q}.x+$ &c. expressed generally in terms of p, q, and x, we may find the series, a cos. of $pz+b$. cos. of $\overline{p+q}.z+$ &c. expressed generally in terms of p, q, and z, and the contrary. For having the sum of the first by writing π for p, z for q, we shall have the sum of the series, a sine of $\pi z+b$ sine of $\overline{\pi+z}.z+$ &c. $=A$, expressed by z, and particular values; in which writing $\frac{\pi x}{q}$ for z, we get A', therefore having A' and Q, we may, by

help of *Cor.* II. find P in terms of p, q, and z, and particular values, namely, the sum of the series a cos. of $pz+b$ cos. of $\overline{p+q}.x$ &c. and in a similar manner the contrary is proved. Q. E. D.

Theorem VII.

If we have the sum of the series, a sine of $pz+b$ sine of $qz+c$ sine of $rz+$ &c. expressed generally by z; we have likewise the sum of the series, a cos. of px. sine of $py+b$ cos. of qx. sine of $qy+c$ cos. of rx sine of ry &c. expressed generally by x and y. And if we have the sum of the series, a cos. of $pz+b$ cos. of $qz+c$ cos. of rz &c. expressed generally by z; we have likewise the sum of the series, a cos. of px. cos. of $py+b$ cos. of qx. cos. of $qy+c$ cos. of rx. cos. of ry &c.; and also the sum of the series, a sine of px. sine of $py+b$ sine of qx. sine of $qy+c$ sine of rx sine of ry &c. expressed generally in terms of x and y.

First; if we have the sum of the series, a sine of $pz+b$ sine of qz &c. expressed in terms of z, by writing $x+y$ in the room of z throughout, we shall have the sum of the series, a sine of $p.\overline{x+y}+b$ sine of $q.\overline{x+y}+c$ sine of $r.\overline{x+y}$ &c. expressed in terms of x and y and in like manner by writing $x-y$ for z we shall have the sum of the series, a sine of $p.\overline{x-y}+b$ sine of $q.\overline{x-y}+c$ sine of $r.\overline{x-y}$ &c. expressed in terms of x and y, therefore the half difference of these two, that is, $a.\frac{\text{sine of } p.\overline{x+y} - \text{sine of } p.\overline{x-y}}{2}+b.\frac{\text{sine of } q.\overline{x+y} - \text{sine of } q.\overline{x-y}}{2}+c.\frac{\text{sine of } r.\overline{x+y} - \text{sine of } r.\overline{x-y}}{2}$ &c. or its equal by lemma No. II, a cos. of px. sine of $py+b$ cos. of qx. sine of $qy+c$ cos. of rx.

sine of ry + &c. will likewise be expressed generally in terms of x and y, which is case the first.

Again, if we have the sum of the series a cos. of $pz + b$ cos. of qz &c., expressed generally by z; by writing $x - y$ throughout for z we shall have the sum of the series a . cos. of $p \,.\, \overline{x-y} + b$ cos. of $q \,.\, \overline{x-y} + c$ cos. of $r \,.\, \overline{x-y}$ &c. expressed generally by x and y and by writing $\overline{x+y}$ for z throughout, we shall have the sum of of the series a cos. of $p \,.\, \overline{x+y} + b$ cos. of $q \,.\, \overline{x+y} + c$ cos. of $r \,.\, \overline{x+y}$ &c. expressed generally in terms of x and y, and consequently the half sum of the two which by lemma No. III. is equal to a cos. of px . cos. of $py + b$. cos. of qx . cos. of $qy + c$ cos. of rx . cos. of ry &c. will be expressed generally in terms of x and y; and the half difference of the two which by lemma No. I. is equal to, a sine of px . sine of $py + b$ sine of qx . sine of $qy + c$ sine of rx . sine of ry &c. will likewise be expressed generally in terms of x and y. Q. E. D.

Corollary. From the sum of the series, a sine of $pz + b$ sine of qz + &c. having obtained the sum of the series, a cos. of of px . sine of $py + b$ cos. of qx . sine of qy &c. if a' be put for a cos. of px, b' for b cos. of qx, c' for c cos. of rx, &c. this series will be reduced to a' sine of $py + b'$ sine of $qy + c'$ sine of ry &c. which is of the first form of this theorem, and consequently from it may be deduced the sum of the series a' cos. of pw . sine of $pv + b'$ cos. of qw . sine of $qv + c$ cos. of rw . sine of rv &c. and therefore its equal the sum of the series a cos. of pw . cos. of px . sine of $pv + b$ cos. of qw . cos. of qx . sine of $qv + c$ cos. of rw . cos. of rx . sine of rv + &c. in terms of v, w, and x, but if a' had been put for a sine of py, b' for

b sine of qy &c.; we should have had the series reduced to the form a' cos. of $py + b'$ cos. of qy &c. which is of the 2d form of the theorem, and consequently from it is deduced the sum of each of the series, 1st. a' cos. of pw . cos. of $pv + b'$ cos. of qw . cos. of qv &c. that is, of the series a cos. of pw . cos. of pv . sine of $px + b$ cos. of qw . cos. of qv . sine of qx &c. in terms of v, w, and x, which is indeed similar in form to the series found by the other substitution; and 2d. the sum of the series a' sine of pw . sine of $pv + b'$ sine of qw . sine of qv &c. or its equal the sum of the series a sine of pw . sine of px . sine of $pv + b$ sine of qw . sine of qx . sine of $qv +$ &c. in terms of v, w, and x. And in a similar manner, from the sum of the series a cos. of $px + b$ cos. of qx &c. having found the sum of the series a cos. of px . cos. of $py + b$ cos. of qx . cos. of qy &c. we may find the sum of the series a cos. of pw . cos. of pv . cos. of $px + b$ cos. of qw . cos. of qv . cos. of $qx +$ &c. in terms of w, v, and x, and likewise the sum of the series a cos. of pw . sine of pv . sine of $px + b$ cos. of qw . sine of qv . sine of qx &c. And in a similar manner also may we proceed by degrees to more complicated cases.

Example 1. Because (from *Example* 1. *Scholium* 2. after *Example* 3. *Theorem* IV.) we have the sum of the r first terms of the series, cos. of $nz +$ cos. of $\overline{n+q}.z +$ cos. of $\overline{n+2q}.z$ &c. $= [\text{sine of } n + \overline{r-\tfrac{1}{2}q}.z - \text{sine of } \overline{n-\tfrac{1}{2}q}.z]$: 2 sine of $\frac{1}{2}qz$: if $x - y$ and $x + y$ be written for z, then the half sum and half difference of the resulting expressions, by case 2 of this theorem, will give the r first terms of the series cos. of nx . cos. of $ny +$ cos. of $\overline{n+q}.x$. cos. of

$\overline{n+q}.y$ &c. $= \frac{\text{sine of } \overline{n+r-\frac{1}{2}q}.\overline{x-y} - \text{sine of } \overline{n-\frac{1}{2}q}.\overline{x-y}}{4 \text{ sine of } \frac{1}{2}q.x-y}$

$+ \frac{\text{sine of } \overline{n+r-\frac{1}{2}q}.\overline{x+y} - \text{sine of } \overline{n-\frac{1}{2}q}.\overline{x+y}}{4 \text{ sine of } \frac{1}{2}q.x+y}$; and the sum of the r first terms of the series sine of nx . sine of ny + sine of $\overline{n+q}.x$. sine of $\overline{n+q}.y$ + sine of $\overline{n+2q}.x$. sine of $\overline{n+2q}.y$ &c. $= \frac{\text{sine of } \overline{n+r-\frac{1}{2}q}.\overline{x-y} - \text{sine of } \overline{n-\frac{1}{2}q}.\overline{x-y}}{4 \text{ sine of } \frac{1}{2}q.x-y} - [\text{sine of } \overline{n+r-\frac{1}{2}.q}.\overline{x+y} - \text{sine of } \overline{n-\frac{1}{2}q}.\overline{x+y}] \div 4 \text{ sine of } \frac{1}{2}q.x+y$.

Cor. If rx and ry be both multiples of the whole circumference the said two values will be equal to 0.

Example 2. Because (from *Cor.* 1. *Example* 2. *Theorem* II.) we have sine of $px - r$ sine of $\overline{p+q}.z + r.\frac{r+1}{2}$ sine of $\overline{p+2q}.z$ &c. $= \frac{\text{sine of } \overline{p-\frac{1}{2}qr}.z}{2 \text{ cos. of } \frac{1}{2}qz}$, we have by this theorem case 1, cos. of px . sine of $py - r$ cos. of $\overline{p+q}.x$. sine of $\overline{p+q}.y + r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.x$. sine of $\overline{p+2q}.y$ &c. $= \frac{\text{sine of } \overline{p-\frac{1}{2}qr}.\overline{x+y}}{2 \text{ . } 2 \text{ cos. of } \frac{1}{2}q.x+y} - \frac{\text{sine of } \overline{p-\frac{1}{2}qr}.\overline{x-y}}{2 \text{ . } 2 \text{ cos. of } \frac{1}{2}q.x-y}$. And because by the same cos. of $px - r$ cos. of $\overline{p+q}.z + r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.z$ &c. $= \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.z}{2 \text{ cos. of } \frac{1}{2}qz}$ we have by case 2 of this theorem cos. of px . cos. of $py - r$ cos. of $\overline{p+q}.x$. cos. of $\overline{p+q}.y + r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.x$. cos. of $\overline{p+2q}.y$ &c. $= \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.\overline{x-y}}{2 \text{ . } 2 \text{ cos. of } \frac{1}{2}q.x-y} + \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.\overline{x+y}}{2 \text{ . } 2 \text{ cos. of } \frac{1}{2}q.x+y}$ and sine of px . sine of $py - r$ sine of $\overline{p+q}.x$. sine of $\overline{p+q}.y$ + &c. $= \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.\overline{x-y}}{2 \text{ . } 2 \text{ cos. of } \frac{1}{2}q.x-y} - \frac{\text{cos. of } \overline{p-\frac{1}{2}qr}.\overline{x+y}}{2 \text{ . } 2 \text{ cos. of } \frac{1}{2}q.x+y}$.

Cor. If in these we either put $qr - p$ or $\frac{1}{2}qr$ in the place of p, we shall get theorems for the rectangles of sine and cosines, rectangles of cosines and the rectangles of sines similar to those of *Cor.* II. and III. (respectively) *Example* 2. *Theorem* II. for the simple sines and cosines.

General Scholium.

It is necessary to obeserve, that there may be particular cases, in the summation of which these methods fail, and which, if not properly considered, may lead to great error, especially when new series are derived from those containing failing cases, by multiplying by fluxions, and finding the fluents of the expressions thence arising. For if the correction should happen to be sought from any of the failing cases, the summation of the new series might not only fail in the failing case of the primary expression, but in every other.

From *Example* 1. *Theorem* I. we have sine of pz + sine of $\overline{p+q}.z$ + sine of $\overline{p+2q}.z$ &c. $= \frac{\text{cos. of } \overline{p-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$; this when $z = 0$, will be sine of 0 + sine of 0 + sine of 0 &c. or $0 + 0 + 0$ &c. $= \frac{\text{cos. of } 0}{2 \text{ sine of } 0} = \frac{1}{2 \text{ sine of } 0}$; that is the sum of a series of noughts infinite, which is absurd. Again, in *Example* 2. *Theorem* I. *Cor.* 1. cos. of nz + cos. of $3nz$ + cos. of $5nz$ &c. $= 0$, therefore if z be taken $= 0$ it will be $1 + 1 + 1$ &c. $= 0$ which ought to be infinite, and in *Cor.* II. z being $= 0$ we have $1 + 1 + 1 + 1$ &c. $= -\frac{1}{2}$.

In order to explain the reason of these absurdities, and to prevent the errors they may produce, it is necessary to con-

sider the subject more minutely, to which purpose *Scholium* II. at the end of *Theorem* IV. will afford great assistance: from that it appears, that the sum of the r first terms of the series cos. of nz + cos. of $\overline{n+q}.z$ + cos. of $\overline{n+2q}.z$ &c. $=$ $\frac{\text{sine of } \overline{n+r-\frac{1}{2}q}.z - \text{sine of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$; and by similar means that the sum of the r first terms of the series sine of nz + sine of $\overline{n+q}.z$ + sine of $\overline{n+2q}z$ + &c. $=$ $\frac{-\text{cos. of } \overline{n+r-\frac{1}{2}q}.z + \text{cos. of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$: now it is plain that if qz were either equal to 0 or a multiple of 360°, sine of $\frac{1}{2}qz$ would be equal to 0, and because r is a whole number, rqz would either be equal to 0 or a multiple of 360°, and consequently the sine of $\overline{n+r-\frac{1}{2}.q}.z =$ sine of $\overline{n-\frac{1}{2}q}.z$ and the cosine of $\overline{n+r-\frac{1}{2}.q}.z =$ cos. of $\overline{n-\frac{1}{2}q}.z$, and therefore the sum of the series, cos. of nz + cos. of $\overline{n+q}.z$ &c. $=$ (when $qz=0$ or some multiple of 360°) - - - - $\frac{\text{sine of } \overline{n-\frac{1}{2}q}.z - \text{sine of } \overline{n-\frac{1}{2}q}.z}{0} = \frac{0}{0}$, and of sine of nz + sine of $\overline{n+q}.z$ &c. $= \frac{0}{0}$ whatever r may, whether finite or infinite. Indeed the determining the value of $\frac{0}{0}$, depends on the value of r; but if qz be any thing but 0 or a multiple of 360°, the value of the sine or cosine of $\overline{n+r-\frac{1}{2}q}.z$ will depend on the value of r, and may then be varied from positive to negative and from negative to positive, by merely increasing r, and consequently when r is infinite, there being no reason for its being positive rather than negative, or negative rather than positive it should be considered 0; and therefore the sum of the infinite series, cos. of nz + cos. of $\overline{n+q}.z$ &c. $=$ $-\frac{\text{sine of } \overline{n-\frac{1}{2}q}.z}{\text{sine of } \frac{1}{2}qz}$ and of sine of nz + sine of $\overline{n+q}.z$ &c. $=$

$\frac{\text{cos. of } n-\frac{1}{2}q.z}{2 \text{ sine of } \frac{1}{2}qz}$ in every case, (the same as in *Example* to *Theorem* I.) except when $qz=0$ or some multiple of 360°; on account of there being something else to be taken into consideration, in that case. Again, it appears by *Example* 2 of the said scholium, that the sum of the r first terms of the series, cos. of nz — cos. of $\overline{n+q}.z+$ cos. of $\overline{n+2q}.z-$ &c. $=$ $\frac{\text{cos. of } n-\frac{1}{2}q.z \pm \text{cos. of } \overline{n+r-\frac{1}{2}q.z}}{2 \text{ cos. of } \frac{1}{2}qz}$, the upper or under sine to be taken according as r is odd or even. And by similar means the sum of the r first terms of the series, sine of nz — sine of $\overline{n+q}.z+$ sine of $\overline{n+2q}.z-$ sine of $\overline{n+3q}.z$ &c. is found $=$ $\frac{\text{sine of } n-\frac{1}{2}q.z \pm \text{sine of } \overline{n+r-\frac{1}{2}q.z}}{2 \text{ cos. of } \frac{1}{2}qz}$, the upper sign to be taken if r is odd but the under if even. Here if $qz=180°$, or any odd multiple thereof, the cosine of $\frac{1}{2}qz$ will be $=0$; and if r be even at the same time the cos. of $\overline{n+r-\frac{1}{2}q.z}$ will be equal to the cosine of $\overline{n-\frac{1}{2}q.z}$ and the sine of $\overline{n+r-\frac{1}{2}q.z}=$ sine of $\overline{n-\frac{1}{2}qz}$; but if r be odd we shall have the cos. of $\overline{n+r-\frac{1}{2}q.z}$ $=-$ cos. of $\overline{n-\frac{1}{2}q.z}$ and sine of $\overline{n+r-\frac{1}{2}q.z}=-$ the sine of $\overline{r-\frac{1}{2}q.z}$; consequently by substituting these values in the above expressions for the sum, due regard being had to the signs, we shall find that, whatever r be, the sum of either series will be expressed by $\frac{0}{0}$: but if z be any other value it appears that $\pm$ the sine or cosine of $\overline{n+r-\frac{1}{2}q.z}$ depends on the value of r, and may be either positive or negative, by varying r; and consequently should as above when r is infinite be considered $=0$. And the sum of the series, sine of

$nz-$ sine of $\overline{n+q.z+}$ &c. will be $=\frac{\text{sine of } n-\frac{1}{2}q.z}{2 \text{ cos. of } \frac{1}{2}qz}$, and of cos. of $nz-$ cos. of $\overline{n+q.z+}$ &c. $=\frac{\text{cos. of } n-\frac{1}{2}q.z}{2 \text{ cos. of } \frac{1}{2}qz}$, as in *Example* 1, *Theorem* II. except when $qz=$ some odd multiple of 180°, something else being in that case to be taken into consideration; and thus are we to reason, in the failing cases of other expressions: but by the common rules for finding the value of an expression when the denominator and numerator vanish, we may find the value even in the failing cases; thus by dividing the fluxion of the numerator by the fluxion of the denominator in the expressions $\frac{-\text{ cos. of } \overline{n+r-\frac{1}{2}q.z+} \text{ cos. of } \overline{n-\frac{1}{2}q.z}}{2 \text{ sine of } \frac{1}{2}qz}$ and $\frac{\text{sine of } \overline{n+r-\frac{1}{2}q.z-} \text{ sine of } \overline{n-\frac{1}{2}q.z}}{2 \text{ sine of } \frac{1}{2}qz}$, and then making $qz=0$, or some multiple of 360°, we shall get simply, 0 for the sum of the r first terms of the series sine of $nz+$ sine of $\overline{n+q.z+}$ &c., and r for the sum of the r first terms of the series cos. of $nz+$ cos. of $\overline{n+q.z+}$ &c. when $z=0$ or some multiple of 360°, that is, 0 for the sum of the r first terms of the series, $0+0+0$ &c. and r for the sum of the r first terms of the series, $1+1+1$ &c. which is self-evident. And thus may we proceed in other expressions when the sum of r terms can be obtained by a general value.

That these things should happen as above described, is likewise evident, from the investigations of the theorems; for in *Theorem* I. for instance, we have $s^x = \pm 2 \overline{\text{ sine of } \frac{1}{2}qz|}^{x}$ or $\pm s.\overline{2 \text{sine of } \frac{1}{2}qz|}^{x}$, x being a positive whole number; therefore if the sine of $\frac{1}{2}qz$ be $=0$, which will happen when $qz=0$ or some multiple of 360°, it is plain that we should have

$s^r = \pm s' \times 0$, or $\pm s \times 0$, and consequently s^r ought likewise to come out equal to 0, and therefore s, would be $= \frac{0}{0}$; and consequently when s^r in that case does not come out $= 0$, it is certain that there must have been something neglected: and to shew how this may happen, we observe that since *Theorem* I. and II. require the differences of the coefficients of every term and the next succeeding term to be taken, it is evident that the last term will have nothing to be taken from, and will consequently remain through every new series; in consequence of which there will be terms of the form Δ. sine or cosine of $\overline{\pi + qr.z}$, (in which r is a whole number and infinite, the number of terms of the series being infinite,) whose coefficient Δ will never be $= 0$ unless the series $a, b, c,$ &c. be converging: these terms are unavoidably omitted, by reason of their place being at an infinite distance, and can consequently never be arrived at; but still unless it be equal to 0, it should not be omitted; which it cannot be unless, either in the above mentioned circumstance of the series $a, b, c,$ &c. being converging, or when the terms of the series of sines or cosines, are continually changing their signs, for different values of r; which it will always do when qz is not equal to 0 or some multiple of 360°; provided the coefficients $a, b, c,$ &c. are all affirmative: and consequently the said terms may be omitted in every such case, there being no reason for taking one sign rather than the other: but if qz were equal to 0 or some multiple of 360°, since Δ. sine or cosine of $\overline{\pi + qr.z}$ will then be simply Δ. sine or cos. of pz, and therefore if the same sign whatever r may be, when $a, b, c, d,$ &c. ---- to Δ, have all the same signs; and consequently cannot

be then neglected unless in the case above mentioned of a, b, c, &c. being converging, in which circumstance it will have no failing case: but had the coefficients a, b, c, d, &c. been alternately $+$ and $-$, the failing case would not happen when $qx=0$ or a multiple of 360°, for then there being no reason for taking Δ of one sign rather than the other, it should therefore be taking equal to 0; but it will happen when sine or cos. of $\overline{w+qr.x}$ is alternately positive and negative, by continually increasing r by 1: for then the coefficients of the terms of the form, sine or cos. of $\overline{w+qr.x}$ being alternately positive and negative; and likewise the terms themselves alternately positive and negative, the whole values resulting from them will have the same determinate sign, and this will be when $qx=180°$ or some multiple thereof. And if a, b, c, d, &c. be positive and negative according to some other law, the failing cases may be found by the like reasoning; which is likewise applicable to the other theorems.

These remarks pave the way to the correction of fluents necessary in the application of the doctrine of fluxions to these series.

1. In *Example* 2, *Theorem* 1. if for nx we write $k+x$, and for q we write 1, we shall have cos. of $\overline{k+x}+$ cos. of $\overline{k+2x}+$ cos. of $\overline{k+3x}$ &c. $=-\frac{\text{sine of }\overline{k+\frac{1}{2}x}}{2\text{ sine of }\frac{1}{2}x}=-\text{ sine of }k.\frac{\text{cos. of }\frac{1}{2}x}{2\text{ sine of }\frac{1}{2}x}-\frac{\text{cos. of }k.\text{ sine of }\frac{1}{2}x}{2\text{ sine of }\frac{1}{2}x}=-\frac{\text{sine of }k.\text{ cos. of }\frac{1}{2}x}{2\text{ sine of }\frac{1}{2}x}-\frac{\text{cos. of }k}{2}$. Multiply both sides by $\dot{x}$, and find the consequent fluents, and we shall have sine of $\overline{k+x}+\frac{\text{sine of }\overline{k+2x}}{2}+\frac{\text{sine of }\overline{k+3x}}{3}$ &c. $=$ fluent of $\left[-\frac{\dot{x}}{2}\text{ sine of }k.\frac{\text{cos. of }\frac{1}{2}x}{\text{sine of }\frac{1}{2}x}-\frac{\text{cos. of }k}{2}\dot{x}\right]$, which because $\frac{\dot{x}}{2}$ cos. of

$\frac{1}{2}x$, is equal to the fluxion of sine of $\frac{1}{2}x$, is equal to the fluent of $\left[-\text{sine of } k \cdot \frac{\text{fluxion of sine of } \frac{1}{2}x}{\text{sine of } \frac{1}{2}x} = \frac{\text{cos. of } k}{2} \cdot \dot{x}\right] = -$ sine of k. log. sine of $\frac{1}{2}x - \frac{\text{cos. of } k}{2} \cdot x +$ a correction: now this correction must not be sought when $x = 0$ or a multiple of 360°: for in that case from what has been just now said, the primary equation fails, or rather there is a supplemental value only then to be prefixed: therefore the easiest method which offers, is when $x = 180°$, we then have the sine of $k + x = -k$, sine of $k + 2x =$ sine of k, sine of $k + 3x = -$ sine of k &c. and sine of $\frac{1}{2}x = 1$, consequently putting Q for $\frac{1}{4}$ of the periphery of a circle whose radius one, the expression will become $-$ sine $k + \frac{\text{sine of } k}{2} - \frac{\text{sine of } k}{3} + \frac{\text{sine of } k}{4}$ &c. or sine of $k \times$ log. of $\frac{1}{2} =$ $-$ cos. of $k \cdot Q +$ correction $\therefore$ correction $=$ sine of k. log. of $2 +$ cos. of $k \cdot Q$, which correction being prefixed we have, sine of $k + x + \frac{\text{sine of } k + x}{2} + \frac{\text{sine of } k + 2x}{3}$ &c. $=$ sine of $k \times$ log. of $\frac{1}{2 \text{ sine of } \frac{1}{2}x} + Q - \frac{x}{2} \times$ cos. of k: which is only true whilst x is between 0 and 360°; for though our primary equation fails only when x is 0 or some multiple of 360°, and is true in every other case, whatever x may be, whether more or less than 360°; still it cannot be so in this derivative equation: for suppose K to be the said supplemental value, which is equal to 0 in every other case but that mentioned above, the derivative expression, will in that case contain the supplement, the fluent of $K \cdot x$ producing a correction which will remain when it is once introduced, though K may afterwards vanish, namely, when x becomes neither 0, nor any multiple of 360°

and therefore every time z becomes, by flowing, any multiple of 360°; K being introduced, it will introduce an additional correction, which will remain afterwards.

If this equation be multiplied by $\dot{z}$ and the fluent be again taken, we shall have $\frac{\text{cos. of } k+z}{1} + \frac{\text{cos. of } k+2z}{4} + \frac{\text{cos. of } k+3z}{9}$ &c. $= -$ fluent of [sine of k . $\dot{z}$ log. of $\frac{1}{2 \text{ sine of } \frac{1}{2}z}$ $+ Q - \frac{\pi}{2}$. $\dot{z}$ cos. of k] which fluent is easily found by infinite series, but if k be $= 0$ we shall have cos. of $k = 1$, and the fluent $= Qz - \frac{z^2}{4}$ independent of the correction, that is cos. of $z + \frac{\text{cos. of } 2z}{2^2} + \frac{\text{cos. of } 3z}{3^2}$ &c. $= A - Qz + \frac{z^2}{4}$, A standing for the correction: if z be $=$ to the arc of 90° or Q, we shall have cos. of $z = 0$, cos. of $2z = -1$, cos. of $3z = 0$, cos. of $4z = 1$ &c. therefore we shall have by substitution $-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \frac{1}{8^2}$ &c. $= A - Q' + \frac{Q'}{4} = A - \frac{3}{4}Q'$, but if in the equation z be taken $= 180°$ or $2Q$, we shall have cos. of $z = -1$, cos. of $2z = +1$, cos. of $3z = -1$, &c. &c. $\therefore -1 + \frac{1}{2^2} - \frac{1}{3^2} +$ &c. $= A - 2Q' + Q'$ or $A - Q'$, which series being the same as the other series when multiplied by $\frac{1}{4}$, we have $A - Q' = \frac{1}{4}A - 2Q' \therefore A = \frac{4}{3}Q' \therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} -$ &c. $= Q' - A = \frac{Q'}{3}$, and cos. of $z + \frac{\text{cos. of } 2z}{4} + \frac{\text{cos. of } 3z}{9}$ &c. $= \frac{2}{3}Q' - Qz + \frac{z^2}{4}$. It is remarkable that this equation is true, not only when the equation from which it is derived is true, but likewise when $z = 0$ or 360° in which that fails, and that the correction might have been sought in those cases had this circumstance been known. Multiply this again by $\dot{z}$, and find the fluent, and we

have sine of $z + \frac{\text{sine of } 2z}{8} + \frac{\text{sine of } 3z}{3^3}$ &c. $= \frac{2}{3}Qz - \frac{Qz^3}{2} + \frac{z^3}{12}$ and this requires no correction whilst z is from 0 to 360 inclusively of both; this is evident at first sight, as we are not now obliged as before to avoid correcting when $z = 0$ or 360°, as the equation from which this is derived does not fail in those cases.

2. If in the equation sine of nz — sine of $n + q \cdot z$ + sine of $\overline{n + 3q}z$ — &c. $= \frac{\text{sine of } \overline{n - \frac{1}{2}q}z}{2 \text{ cos. of } \frac{1}{2}qz}$, failing when $qz = 180°$ or an odd multiple thereof, we put $n = 1$, $q = 2$ we have sine of z — sine of $3z$ + sine of $5z$ &c. $= 0$, failing when $z = 90°$ or any odd multiple thereof; if we multiply this by z and find the fluent we shall have cos. of $z - \frac{\text{cos. of } 3z}{3} + \frac{\text{cos. of } 5z}{5}$ — &c. = correction, which must not be sought when $z = 90°$, or odd multiple thereof; if it be sought when $z = 0$, we shall have it $= 1 - \frac{1}{3} + \frac{1}{5}$ &c. $= \frac{1}{2}Q$, which will answer whilst z is exclusively from — 90° to + 90°. If the correction had been sought when $z = 180°$ we should have it $= -1 + \frac{1}{3} - \frac{1}{5}$ &c. $= -\frac{1}{2}Q$, answering whilst z is from 90° to 270°.

3. Again, from *Cor.* 1. *Example* 2. *Theorem* 1. cos. of z + cos. of $3z$ + cos. of $5z$ + &c. is equal to 0, failing (from above) when $qz = 0$ or a multiple of 360°, and therefore when $z = 0$ or a multiple of 180°: if we multiply by the z and take the fluent we have sine of $z + \frac{\text{sine of } 3z}{3} + \frac{\text{sine of } 5z}{5}$ &c. = correction, which should not be sought when $z = 0$ or any multiple of 180, when $z = 90°$ it becomes sine of $90° + \frac{\text{sine of } 3 \times 90°}{3} + \frac{\text{sine of } 5 \times 90°}{5}$ &c. that is $1 - \frac{1}{3} + \frac{1}{5}$ &c. or its equal $\frac{Q}{2}$ for the

correction, the same as LANDEN finds, this is true whilst z is exclusively between 0 and 180°.

4. In *Cor.* II. *Example* 1. *Theorem* I. p being $= 1$ we have sine of z + sine of $3z$ + sine of $5z$ &c. $= \frac{1}{2 \text{ sine of } z}$ failing (from above) when qz or $2z = 0$, or a multiple of 360°, and therefore when $z = 0$, or a multiple of 180°; if we multiply this by $\dot{z}$ and find the fluent we have, because $\frac{\dot{z}}{2 \text{ sine of } z}$ (by putting y for the sine of z) $= \frac{y^{-1}\dot{y}}{2\sqrt{1-y^2}} = \frac{y^{-1}\dot{y}}{2\sqrt{y^2-1}} = -\frac{\dot{x}}{2\sqrt{x^2-1}}$ (x being put for $\frac{1}{y}$) whose fluent is $= -\frac{1}{2}\log. \text{ of } x + \sqrt{x^2-1} = -\frac{1}{2}\log. \text{ of } \frac{1+\sqrt{1-y^2}}{y}$, consequently cos. of $z + \frac{\text{cos. of } 3z}{3} + \frac{\text{cos. of } 5z}{5} = \frac{1}{2}\log. \text{ of } \frac{1+\sqrt{1-y^2}}{y}$ + correction, which being sought when $z = 90°$ and consequently $y = 1$ and the cosines of z, of $3z$, of $5z$, &c. $= 0$, we have it equal to 0; and this has no failing case since it will not fail when $z = 0$ or any multiple of 180° in which primitive equation does. If z be $= 45°$ we shall have cos. of $z = \sqrt{\frac{1}{2}}$, cos. of $3z = -\sqrt{\frac{1}{2}}$, cos. of $5z = -\sqrt{\frac{1}{2}}$, cos. of $7z = +\sqrt{\frac{1}{2}}$, cos. of $9z = +\sqrt{\frac{1}{2}}$, &c. therefore $\sqrt{\frac{1}{2}} \times 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} -$ &c. or $\sqrt{\frac{1}{2}} \times 1 - \frac{8}{3.5} + \frac{16}{7.9} - \frac{24}{11.13}$ &c. $= \frac{1}{2}\log. \text{ of } \frac{1+\sqrt{2}}{\sqrt{\frac{1}{2}}} \therefore 1 - \frac{8}{3.5} + \frac{16}{7.9}$ &c. $= \sqrt{\frac{1}{2}}\log. \text{ of } \frac{1+\sqrt{2}}{\sqrt{\frac{1}{2}}} = \sqrt{\frac{1}{2}}\log. \text{ of } \sqrt{2}+1 \therefore \frac{1}{3.5} - \frac{2}{7.9} + \frac{3}{11.13}$ &c. $= \frac{1}{8} + \frac{\sqrt{\frac{1}{2}}}{8}\log. \text{ of } \sqrt{2}+1$.

5. By *Theorem* I. *Example* 6, we have the sum of the series sine of $pz + g$ sine of $\overline{p+q}.z + g^2$ sine of $\overline{p+2q}.z +$ &c.

$= \frac{-g \text{ sine of } \overline{p-q}\,.\,z + \text{sine of } pz}{g^2+1-2g \text{ cos. of } qz}$, if this should have a failing case, it will be, by this scholium, when $qz = 0$ or some multiple of 360° provided g be affirmative; but if g be negative, it will be when qz is some odd multiple of 180°; a similar expression to this is given by Mr. LANDEN by his method.

If p be $= q = 1$ we shall have, sine of $z + g$ sine of $2z + g^2$ sine of $3z$ &c. $= \frac{\text{sine of } z}{g^2+1-2g \text{ cos. of } z}$, if we now multiply by $\dot{z}$ calling the cosine of z, x and find the fluent, we shall have cos. of $z + \frac{g \text{ cos. of } 2z}{2} + \frac{g^2 \text{ cos. of } 3z}{3}$ &c. $= \frac{1}{2g}$. log. of $1+g^2-2gx$ which has no failing case.

6. According to this *General Scholium*, *Example* 2. to *Theorem* II. has failing cases in the investigation, unless the series $1, r, r.\frac{r+1}{2}$ &c. converge; thus those in the *Corollaries* I. II. and III. when qz is any odd multiple of 180°. By bringing both series to one side in the equations in *Cor.* II. we have $\overline{\text{sine of } qr-p\,.\,z + \text{sine of } pz} - r \times \overline{\text{sine of } q\,.\,r+1-p\,.\,z + \text{sine of } p+q\,.\,z}$ + &c. $= 0$, and $\overline{\text{cos. of } qr-p\,.\,z} - \text{cos. of } pz - r \times \overline{\text{cos. of } q\,.\,r+1-p\,.\,z} - \text{cos. of } \overline{p+q\,.\,z}$ + &c. $= 0$. Multiply them both by $\dot{z}$ and take the correct fluents when $z = 0$, and we get $\frac{\text{cos. of } \overline{qr-p\,.\,z}}{qr-p} + \frac{\text{cos. of } pz}{p} - r.\frac{\text{cos. of } \overline{q\,.\,r+1-p\,.\,z}}{q\,.\,r+1-p} + \frac{\text{cos. of } \overline{p+q\,.\,z}}{p+q} +$ $r.\frac{r+1}{2}.\frac{\text{cos. of } \overline{q\,.\,r+2-p\,.\,z}}{q\,.\,r+2-p} + \frac{\text{cos. of } \overline{p+2q\,.\,z}}{p+2q}$ &c. $= M$, and $\frac{\text{sine of } \overline{qr-p\,.\,z}}{qr-p} - \frac{\text{sine of } pz}{p} - r.\frac{\text{sine of } \overline{q\,.\,r+1-p\,.\,z}}{q\,.\,r+1-p} - \frac{\text{sine of } \overline{p+q\,.\,z}}{p+q}$ + &c. $= 0$, M being put for $\frac{1}{qr-p} + \frac{1}{p} - r.\frac{1}{q\,.\,r+1-p} + \frac{1}{p+q}$ $+ r.\frac{r+1}{2}.\frac{1}{q\,.\,r+2-p} + \frac{1}{p+2q}$ &c. $= \frac{qr}{p\,.\,qr-p} - r.\frac{r+2\,.\,q}{p+q\,.\,q\,.\,r+1-p}$

$+ r.\frac{r+1}{2}.\frac{r+4.q}{p+2q.q.r+2-p}$ &c., if we now multiply the first of these by sine of $\frac{1}{2}qr-p.z$, and the second by cos. of $\frac{1}{2}qr-p.z$, and take the difference we have

$\frac{\text{sine of } \frac{1}{2}qr-p.z \times \text{cos. of } qr-pz - \text{cos. of } \frac{1}{2}qr-p.z \times \text{sine of } qr-pz}{qr-p} + [\text{sine of } \frac{1}{2}qr-p.z \text{ . cos. of } pz + \text{cos. of } \frac{1}{2}qr-pz \text{ . sine of } pz] \div p - \frac{r}{q.r+1-p} \times [\text{sine of } \frac{1}{2}qr-pz \text{ . cos. of } r+1.q-p.z - \text{cos. of } \frac{1}{2}qr-p.z \text{ . sine of } r+1.q-p.z] - \frac{r}{p+q} \times [\text{sine of } \frac{1}{2}qr-p.z \text{ . cos. of } p+q.z + \text{cos. of } \frac{1}{2}qr-p.z \times \text{sine of } p+q.z]$ &c.

which by trigonometry is reducible to $-\frac{\text{sine of } \frac{1}{2}qr.z}{qr-p} + \frac{\text{sine of } \frac{1}{2}qrz}{p} + r\frac{\text{sine of } q.\frac{1}{2}r+1.z}{q.r+1-p} - r\frac{\text{sine of } q.\frac{1}{2}r+1.z}{p+q}$ &c. $=$ M . sine of $\frac{1}{2}qr-pz$; or $\frac{\text{sine of } \frac{1}{2}qrz}{p.qr-p} - r\frac{\text{sine of } q.\frac{1}{2}r+1.z}{p+q.q.r+1-p} = \text{M}.\frac{\text{sine of } \frac{1}{2}qr-p.z}{qr-2p}$,

the same as LANDEN finds, page 83, Mem. 5. We may farther add, that when series are obtained from others having failing cases by substitution, as in *Scholium* III. to *Theorem* I. or as in *Theorem* V. and VI. regard should be had to those failing cases, according to the manner of substitution, in order to find the failing cases in the new series. We might proceed to many more examples, in finding the sums of new series from others multiplied by fluxions, or we might give examples of finding the sums of new series by throwing others into fluxions: but my chief object in these latter examples was to obviate any difficulty that might appear in choosing the cases for the correction of the fluents. There are other inferences to be drawn, which I may perhaps consider at some future period.

VIII. *An Account of a small Lobe of the human prostate Gland, which has not before been taken Notice of by Anatomists.* By Everard Home, *Esq. F. R. S.*

Read February 20, 1806.

Discoveries in the anatomy of the human body have been ever considered by this learned Society as deserving a place in the Philosophical Transactions: in the present improved state of our knowledge of this subject, a small addition to it cannot fail of being acceptable, since after the long continued labours of so many acute observers, such only can be expected, and even those are rarely to be acquired.

The subject of the present Paper is a portion of a gland, which from the obscurity of its situation has hitherto escaped observation: and were it not for the change produced in it by disease, which enlarges it so much that it sometimes completely shuts up the canal, by which the urine ought to pass, it would be little deserving of attention: but when this important effect is considered, the part itself becomes an object of very serious interest.

In stating the circumstances, which led to the present investigation, it may be necessary to mention that the prostate gland is liable in the latter period of life to enlarge: and when it does so there is frequently a nipple-like projection, which rises up and forms tumours of very different sizes in

the cavity of the bladder. These tumours, as they obstruct the passage of the urine, have attracted the attention of all anatomical surgeons, from the time of MORGAGNI to the present day. Their appearance has been accurately described, and specimens of them in different degrees of enlargement are preserved in every collection of morbid parts. The attention of surgeons has been naturally called to what is of the greatest consideration, the appearances they put on, and the symptoms they produce: but the particular circumstances in the natural conformation of the gland, which dispose it to form these tumours, have never been examined. MORGAGNI says, "These caruncles were found to grow out in the very "middle of the upper and internal posterior circumference of "the gland; but whether these things happened by chance or "otherwise future observations will shew."*

From these expressions, it is evident that MORGAGNI had no idea that there was any conformation of the prostate gland, that could account for this tumour, and believed that it arose from the surface of the body of the gland.

Mr. HUNTER, in treating of the enlargement of the prostate gland, says, "From the situation of the gland, which is prin-"cipally on the two sides of the canal, and but little if at all "on the fore part, as also very little on the posterior side, "when it swells it can only be laterally; whereby it presses

* Si ea, quæ ex Sepulchreto exempla indicavimus, et id, quod supra ex Valsalva attulimus, et nostra omnia attentè inspicias, cuncta in senibus fuisse animadvertes: ita nostra omnia, in quibus caruncularum initium fuit, hanc in medio ipso posteriori interni summique glandulæ ambitûs excrescentem obtulisse: casune hæc cuncta, an secùs, futuræ ostendent observationes. MORGAGNI *de Sed. et Caus. Morb.* lib. iii. epist. 41, A. 19.

"the two sides of the canal together, and at the same time "stretches it from the anterior edge or side to the posterior, "so that the canal instead of being round, is flattened into a "narrow groove. Sometimes the gland swells more on one "side than the other, which makes an obliquity in the canal "passing through it.

"Besides this effect of the lateral parts swelling, a small "portion of it, which lies behind the very beginning of the "urethra swells forward like a point, as it were, into the "bladder; acting like a valve to the mouth of the urethra, "which can be seen, even when the swelling is not consi- "derable, by looking on the mouth of the urethra, from the "cavity of the bladder, in the dead body. It sometimes in- "creases so much, as to form a tumour projecting into the "cavity of the bladder some inches." *

From the first paragraph it is evident that Mr. HUNTER was unacquainted with this lobe; and in the second we see that his knowledge of the disease led him to conclude, that in the natural state of the gland there was a portion of it in this situation: but neither at that time, nor at any future period of his life, did he prosecute the inquiry.

Although a great part of my time has been for many years occupied in attending patients labouring under complaints of the bladder and urethra, and my opportunities of examining these parts after death have been very frequent, my attention has been always so much employed on the modes of emptying the bladder, (an operation, which in many cases is attended with considerable difficulty,) that it never occurred to me to institute an inquiry for the purpose of attaining an accurate

* HUNTER on the Venereal Disease, page 169.

knowledge of the extent of the disease until the month of December, 1805.

At that time my attention was directed to this subject by the following circumstances. In the examination of the prostate gland of an elderly person, who had died in consequence of this part having been diseased, the nipple-like process was found very prominent, and a bridle, nearly ¼ of an inch in breadth, extended from the middle line of the tumour to the bulb of the urethra, where it insensibly disappeared. The usual rounded projection of the *caput gallinaginis* was not visible: it had wasted away, and the remains were concealed in the fold forming this bridle, which at that part was not thicker than at any other. The space between the tumour in the bladder, and the bulb of the urethra was unusually short, which is the reverse of what is commonly met with in old men; so that this bridle appeared to have drawn the bulb towards the tumour, and shortened the membranous part of the canal.

As this was an unusual appearance, it led me to consider it with attention, and to ask if other anatomists had noticed it; which as far as my inquiries have gone has not been the case. The bridle had evidently been formed by the membrane of the bladder adhering firmly to that part of the prostate gland composing the tumour, which it consequently followed in its future increase, and drew up after it the membrane of the urethra. In this way the fold had in time become nearly ¼ of an inch broad, and was continued of the same breadth to the bulb, where the lining of the urethra being more attached to the surrounding parts, it did not admit of being drawn up.

This appearance of a bridle is more or less met with in all the cases, in which the nipple-formed process occurs, but in so much smaller a degree, and not continued beyond the caput gallinaginis, that it never before led me to pay attention to it.

To satisfy myself how this tumour was formed, it became necessary to examine the prostate gland in its natural state; and ascertain whether there is any part sufficiently detached to move independent of the rest of the gland, and explain the appearances which had been met with in this particular case.

My professional avocations not affording time to make the dissections requisite for this purpose, Mr. BRODIE, Demonstrator of Anatomy to Mr. WILSON, Teacher of Anatomy, in Windmill-street, whose knowledge of the subject fitted him for the task, and whose zeal for the improvement of his profession made him willingly undertake it, gave me his assistance, and took the whole of that labour on himself.

While dissecting the parts for this purpose, the urinary bladder was distended with water, and the surfaces of the prostate gland, vesiculæ seminales, and vasa deferentia, were fairly exposed. This being done, the vasa deferentia, and vesiculæ seminales were carefully dissected off from the bladder, without removing any other part. These were turned down upon the body of the prostate gland. An accurate dissection was then made of the circumference of the two posterior portions of the prostate gland, and the space between them was particularly examined. In doing this a small rounded substance was discovered, so much detached that it seemed a distinct gland, and so nearly resembling COWPER's glands in size and shape, as they appeared in the same subject, in which they were unusually large, that it appeared to be a gland of

that kind. It could not however be satisfactorily separated from the prostate gland, nor could any distinct duct be found leading into the bladder.

A similar examination was made of this part in five different subjects. The appearance was not exactly the same in any two of them. In one there was no apparent glandular substance, but a mass of condensed cellular membrane: this, however, on being cut into, differed from the surrounding fat. In another there was a lobe blended laterally with the sides of the prostate gland. These facts are mentioned in proof of its not being always of the same size nor having exactly the same appearance; this is found also to be the case with COWPER's glands, they are sometimes large and distinct; in other subjects are scarcely to be detected, and in others again are in all the intermediate states. The most distinct and natural appearance of this part was in a healthy subject 25 years of age, of which the following is an account. On turning off the vasa deferentia and vesiculæ seminales, exactly in the middle of the sulcus, between the two posterior portions of the prostate gland, there was a rounded prominent body, the base of which adhered to the coats of the bladder. It was imbedded not only between the vasa deferentia and the bladder, but also in some measure between the lateral portions of the prostate gland and the bladder, since they were in part spread over it, so as to prevent its circumference from being seen, and they adhered so closely as to require dissection to remove them; nor could this be done beyond a certain extent, after which the same substance was continued from the one to the other. This proved it to be a lobe of the prostate gland, the middle of which had a rounded form, united to the gland at the base

next the bladder, but rendered a separate lobe by two fissures on its opposite surface. Its ducts passed directly through the coats of the bladder, on which it lay, and opened immediately behind the verumontanum. By means of this lobe a circular aperture is formed in the prostate gland, which gives passage to the vasa deferentia.

The appearance of this lobe has been since examined in a subject 84 years of age, and it was found still larger and more distinct. A representation of it is given in the annexed Plate.

Previous to this investigation it was not known to me that any distinct portion of the prostate gland was situated between the vasa deferentia and the bladder. These ducts were considered to pass in the sulcus between its two posterior portions, in close contact with the body of the gland. This account corresponds also with the description given by WINSLOW and HALLER; it is however now proved to be erroneous. It is not in my power to determine whether all the anatomists of the present day have fallen into this error in the same degree with myself; but none of them have pointed out this lobe; and therefore, in whatever way they have described the vasa deferentia to pass into the bladder, they have neither anticipated nor thrown any light on the present inquiry.* HALLER says expressly, that "the prostate gland has no lobular appearance," and the anxiety which all anatomists have to improve their art, would have led them to correct this error, had they discerned that it was one.

This newly acquired anatomical fact enables us very clearly

* Glandula, aut certe cellulosum compactum corpus, quod prostata dicitur. P. 464. Fabricæ illarum non eo magis glandulosa simplicis similis, cujus cæca essent aliqua, neque composita; neque enim in lobulos recte discedit. P. 465. *Elem. Physiologiæ Corporis Humani, Autore* ALBERT. HALLER. Tom. VII.

to understand the nature of a disease, which it was not possible we could have a correct idea of, when we were ignorant of the existence of the part in which it takes place. It not only explains the situation of the tumour, the want of connection with the body of the gland, and the narrowness of its base where that is met with, but it solves what has ever appeared to me the greatest difficulty, how it should protrude into the cavity of the bladder. This arises from the hard substance of the coats of the vasa deferentia being in close contact, and bound down upon this lobe, so that from its first enlargement it must immediately press up the inner membrane of the bladder, which can make very little resistance.

This lobe of the prostate gland, from its situation and connection with the vasa deferentia, is liable to many causes of swelling, which the body of the gland itself is free from; for every irritation upon the seminal vessels or their orifices may be communicated to it by continuity of parts: and aged men, from an ignorance of these facts, are too often, through imprudence, producing an excitement in those vessels which the parts are unable to support; and when this is long continued, inflammation becomes the consequence, which cannot take place to any degree without being communicated to this lobe, and producing an enlargement of it.

Every violent effort which is made to empty the urinary bladder produces an unusual pressure against this lobe, by which it may be injured. There is also much reason for believing, that the diseased state of the lateral parts of the gland, so very commonly met with in the latter period of life, has its origin in this particular lobe; since in most of the cases of a diseased state of the gland, which have come under my observation after death, this lobe has been enlarged in a

greater degree in proportion to its size than any other part; and in some instances the enlargement of it has been very great, while it appeared to be only beginning in the lateral portions.

Difficulty in passing the urine is a symptom, which comes on very early in diseases of the prostate gland, and arises entirely from this lobe being increased in size, since any enlargement in the lateral portions of the gland widens the canal instead of diminishing it, and they do not require much force to separate them; but the least increase of this lobe tends to shut it up.

The enlargement of this lobe produces an effect which is not generally known, and leads medical practitioners into an error respecting the nature of the complaint. The orifice of the urinary bladder, which is the lowest part in the natural state, is raised up in proportion to the increase of this lobe; so that none of the contents below that level can be expelled, although whatever is above it is allowed with more or less difficulty to pass out. In this way the person never evacuates more than one half or one third of the urine contained in the bladder; but as the water which comes away passes in a stream, and the quantity voided in 24 hours is sufficient, no suspicion is entertained of the cause of the frequency and distress in passing it, and the symptoms are referred to an irritable state of the coats of the bladder. It is only by drawing off the urine through a catheter that the disease in this lobe can be ascertained; as in that way alone the quantity of urine which is retained can be determined. No examination *per anum* can give the surgeon any information on this subject; since the posterior surface of the vasa deferentia only is to be felt, if the finger should reach so far; and yet it is in this way that practitioners in general pretend to judge of the greater or

less degree of the disease, although that portion of the gland, which forms the most important part of the complaint is wholly out of their reach.

The least projection of this lobe into the bladder stretches the internal membrane of that viscus which passes over it, keeps it in a state of irritation, and makes it liable to be grasped by the action of the sphincter muscle in expelling the last drops of urine, so as to give the patient excruciating pain. When it is more enlarged these symptoms go off.

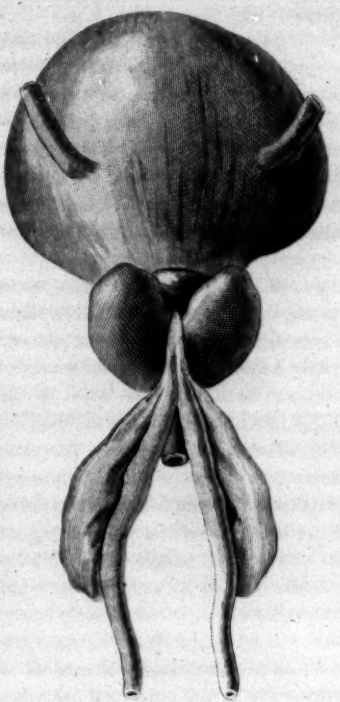
From these observations it appears, that this small lobe of the prostate gland, which has been overlooked, is from the situation and the circumstances, in which it is placed more liable to become diseased than any other part of the gland, and produces symptoms of danger and distress peculiar to itself, which have been hitherto supposed to arise from the body of the gland becoming enlarged.

To enter further into the effects of disease on this lobe would be improper on the present occasion; but not to have noticed them at all would have been equally so, since the only importance, that can be attached to the facts, which have been brought forward in this Paper, arises from the light they throw on the diseases of the prostate gland.

EXPLANATION OF THE PLATE. (Plate III.)

A posterior view of the outside of the bladder and prostate gland.

The vesiculæ seminales and vasa deferentia are dissected off and turned forwards, to shew the newly discovered lobe, which lies between them and the bladder. The two posterior parts of the lateral portions are spread open to expose the lobe lying between them.



IX. *On the Quantity and Velocity of the Solar Motion.* By William Herschel, *LL. D. F. R. S.*

Read February 27, 1806.

The direction of the solar motion having been sufficiently ascertained in the first part of this Paper,* we shall now resume the subject, and proceed to an inquiry about its velocity.

The proper motions, when reduced to one direction, have been called quantities, to distinguish them from the velocities required in the moving stars to produce those motions. It will be necessary to keep up the same distinction with respect to the velocity of the solar motion; for till we are better acquainted with the parallax of the earth's orbit, we can only come to a knowledge of the extent of the arch which this motion would be seen to describe in a given time, when seen from a star of the first magnitude placed at right angles to the motion. There is, however, a considerable difference between the velocity of the solar motion and that of a star; for at a given distance, when the quantity of the solar motion is known its velocity will also be known, and every approximation towards a knowledge of the distance of a star of the first magnitude will be an approximation towards the knowledge of the real solar velocity; but with a star it will be otherwise; for though the situation of the plane in which it moves is

* Phil. Trans. for 1805, page 231.

given, the angle of the direction of its motion with the visual ray will still remain unknown.

As hitherto we have consulted only those proper motions which have a marked tendency to a parallactic centre, we ought now, when the question is to determine the velocity of the solar motion, to have in view the real motion of every star whose apparent motion we know; for as it would not be proper to assign a motion to the sun, either much greater or much less than any real motion which may be found to exist in some star or other, it follows that a general review of proper motions ought to be made before we can impartially fix on the solar velocity; but as trials with a number of stars would be attended with considerable inconvenience, I shall use only our former six in laying down the method that will be followed with all the rest.

Proportional Distance of the Stars.

We are now come to a point no less difficult than essential to be determined. Neither the parallactic nor real motion of a star can be ascertained till its relative distance is fixed upon. In attempting to do this it will not be satisfactory to divide the stars into a few magnitudes, and suppose *these* to represent the relative distances we require. There are not perhaps among all the stars of the heavens any two that are exactly at the same distance from us; much less can we admit that the stars which we call of the first magnitude are equally distant from the sun. And indeed, if the brightness of the stars is admitted as a criterion by which we are to arrange them, it is perfectly evident that all those of the first magnitude must differ as much in distance as they certainly do in

lustre: yet imperfect as this may be, it is at present the only rule we have to go by.

The relative brightness of our six stars, may be expressed as follows: Sirius —; Arcturus —; Capella —; Lyra —; Aldebaran —; Procyon —.

The notations here used are those which have been explained in my first Catalogue of the relative Brightness of the Stars;* but to denominate the magnitudes of these six stars so that they may with some probability represent the distances at which we should place them according to their relative brightness, I must introduce a more minute subdivision than has been commonly admitted, by using fractional distinctions, and propose the following arrangement.

Table VIII.

Proportional Distances of Stars.

Sirius	1,00	Lyra	1,30
Arcturus	1,20	Aldebaran	1,40
Capella	1,25	Procyon	1,40

The interval between Sirius and Arcturus is here made very considerable; but whoever will attentively compare together the lustre of these two stars, when they are at an equal altitude must allow that the difference in their brightness is fully sufficient to justify the above arrangement.

The order of the other four stars is partly a consequence of the distance at which Arcturus is placed, and of the comparative lustre of these stars such as it has been estimated by observations. But if it should hereafter appear that other

* Phil. Trans. for 1796, page 189.

more exact estimations ought to be substituted for them, the method I have pursued will equally stand good with such alterations. I have tried all the known, and many new ways of measuring the comparative light of the stars, and though I have not yet found one that will give a satisfactory result, it may still be possible to discover some method of mensuration preferable to the foregoing estimations, which are only the result of repeated and accurate comparisons by the eye. Whenever we are furnished with more authentic data the calculations may then be repeated with improved accuracy.

Effect of the Increase and Decrease of the Solar Motion, and Conditions to be observed in the Investigation of its Quantity.

The following Table, in which the 2d, 4th, and 5th columns contain the sides of the parallactic triangle, is calculated with a view to show that an increase or decrease of the solar motion will have a contrary effect upon the required real motions of different stars; and as we are to regulate the solar velocity by these real motions, an attention to this circumstance will point out the stars which are to be selected for our purpose.

Table IX.

Stars and relative Distances.	Apparent Motion.	Solar Motion.	Parallactic Motion.	Real Motion.	Velocities.
			"	"	
		1,0	0,67768	+ 0,46518	465175
Sirius	1",11528	1,5	1,01652	+ 0,21701	217007
1,00		2,0	1,35536	— 0,32776	327755
		1,0	0,53579	+ 1,57389	1888670
Arcturus	2",08718	1,5	0,80368	+ 1,30478	1565735
1,20		2,0	1,07158	+ 1,01561	1218736
		1,0	0,79593	— 0,42159	526987
Capella	0",46374	1,5	1,19390	— 0,79637	995465
1,25		2,0	1,59186	— 1,18662	1483270
		1,0	0,32542	— 0,47065	611839
Lyra	0",32435	1,5	0,48812	— 0,59923	778995
1,80		2,0	0,65083	— 0,74185	963750
		1,0	0,63117	— 0,53208	744918
Aldebaran	0",12841	1,5	0,97676	— 0,85737	1200324
1,40		2,0	1,30234	— 1,18283	1655967
		1,0	0,66394	+ 0,59548	833665
Procyon	1",23941	1,5	0,99591	+ 0,30731	430297
1,40		2,0	1,32788	— 0,23885	327890

The real motion of Arcturus contained in the 5th column compared with that of Aldebaran, shows that when the solar motion is increased from 1,0 to 1,5 and to 2",0 the real motion of Arcturus will be gradually diminished from 1,57 to 1,30 and to 1",02, while that of Aldebaran undergoes a con-

trary change from 0,58 to 0,86 and to 1″,18. We may also notice that Capella and Aldebaran, which have a negative sign prefixed to their real motions when the solar motion is 1″,0 are affected differently from Arcturus, Sirius, and Procyon, which have a positive sign; and that even the motions of the two last become negative when the solar motion is increased beyond a certain point. It may be easily understood that the motion of Arcturus itself would become negative were we to increase the solar motion till the parallactic motion of this star should exceed its apparent motion.

From these considerations it appears, that a certain equalization, or approach to equality may be obtained between the motions of the stars, or between that of the sun and any one of them selected for the purpose; for instance, the motions of Arcturus and Aldebaran being contrary to each other, may be made perfectly equal by supposing the sun's annual motion to be 1″,8595. For then we shall have the real annual motion of Arcturus towards the parallactic centre 1″,091, and that of Aldebaran towards the opposite part of the heavens, in which the solar apex is placed, will be 1″,091 likewise; the first in a direction 55° 29′ 39″ south-preceding, the latter 88° 16′ 31″ north-following their respective parallels; and a composition of these motions with the parallactic ones arising from the given solar motion, will produce the apparent motions of these stars which have been established by observation. But since Arcturus, by the hypothesis which has been adopted in Table VIII. is a nearer star than Aldebaran, the velocities of the real motions, describing these equal arches will be 1309109 in the former and 1297780 in the latter. And it is not the arches but these velocities that

must be equalized. Therefore, in order to have this required equality, let the solar motion be $1''{,}718865$ then will a velocity of 1399478 in Arcturus, and 1899842 in Aldebaran, which are sufficiently equal, occasion such angular real motions in the two stars as will bring them, when compounded with their parallactic motions, to the apparent places in which we find them by observation.

Before we proceed, it will be proper to obviate a remark that may be made against this way of equalization or approach to equality. We have said that the calculated velocities are such as would be true if the stars were at the assumed distances, and if their real motions were performed in lines at right angles to the visual ray; to which it may here be objected that the last of these assumptions is so far from having any proof in its favour that even the highest probability is against it. We may admit the truth of what the objection states, without apprehending that any error could arise on that account, if the solar motion were determined by this method. For if the stars do not move at right angles to the visual ray, their real velocity will exceed the calculated one; so that in the first place we should certainly have the minimum of their velocities: and if we were obliged, for want of data to leave the other limit of the motion unascertained, it must be allowed to be a considerable point gained if we could show what is likely to be the least velocity of the solar motion; but a more satisfactory defence of the method is, that if we were to assume a mean of all the angular deviations from the perpendicular to the visual ray that may take place in the directions of the real motions of the stars, the only position we could fix upon as a mean would be an

inclination of 45 degrees. For in this case the chance of a greater or smaller deviation would be equal; and when a number of stars are taken, the deviations either way might then be supposed to compensate each other; but what is chiefly to our purpose, not only the angle of 45 degrees, but also any other, that might be fixed upon as a proper one to represent the mean quantity of sidereal motions, would lead exactly to the same result of the solar velocity to be investigated. For if the velocities of any two stars were equalized, when their motions are supposed to be perpendicular to the visual ray, they would be as much so when they make any other given angle with it; and it is the equalization or approach to equality and not the quantity of the velocities that is the spirit of this method. I have only to add, that an equalization of the solar motion with that of any star selected for the purpose may be had by a direct method of calculation, and will therefore be of great use in settling the rate of the motion to be determined.

It must be evident from what has been said, that a certain mean rate, or middle rank, should be assigned to the motion of the sun, unless very sufficient reasons should induce us to depart from this condition. To obtain this end must consequently be our principal aim; and if we can at the same time bring the sidereal motions to a greater equality among each other, it will certainly be a very proper secondary consideration.

There are two ways of taking a mean of the sidereal motions, one of them may be called the rate and the other the rank. For instance, a number equal to the mean rate of the six numbers, 2, 6, 13, 15, 17, 19, would be 12; but one

that should hold a middle rank between the three highest and three lowest of the six would be 14. In assigning the rate of the solar motion it appears to be most eligible that it should hold a middle rank among the sidereal velocities. We shall however find that nearly the same result will be obtained from either of the methods.

With respect to our second consideration, we may see that it also admits of a certain modification by the choice of the solar motion; for in Table IX. when this motion is 1″,5 the velocity of Arcturus 1565735, will exceed that of Sirius, 217007, more than seven times; whereas a solar motion of 1″ will give us the proportional velocities of these stars as 188867 to 465174; and the former will then exceed the latter only four times.

Calculations for drawing Figures that will represent the observed Motions of the Stars.

The necessary calculations for investigating the solar motion are of considerable extent, and may be divided into two classes, the first of which will remain unaltered whatsoever be the solar motion under examination, while the other must be adjusted to every change that may be required.

The direction of the sun remaining as it has been settled in the first part of this Paper, the permanent computation of each star will contain the annual quantity of the observed or apparent motion, its direction with the parallel of the star, its direction with the parallactic motion, and its velocity. The changeable part will consist of the angular quantity of the real motion, the parallactic direction of this motion, and its velocity.

Before we can make a calculation of the required velocities,

we must fix upon the probable relative distance of the rest of the stars, in the same manner as we have done with the first six. In this I have thought it advisable to distinguish the stars that, from their lustre, may be called principal, and have limited their extent to the brightest of the second magnitude, on account of the uncertainty which still remains about their progressive distances. For though it appears reasonable to allow that the bright stars of the second magnitude may be twice as far from us as those of the first, it will admit of some doubt whether this rule ought to be strictly followed up to the 3d, 4th, 5th, and 6th magnitude; especially when it is not easy to ascertain the boundaries which should limit the magnitudes of very small stars.

The number of these principal stars is 24. The remaining 12 are also arranged by admitting that their magnitudes express their relative distances; and notwithstanding the doubtfulness we have noticed, their testimony with respect to the proper quantity of a solar motion, though it should be received with some diffidence, must not be neglected; some considerable alteration in their supposed distances, however, would have but little effect upon the conclusions intended to be drawn from their velocities.

The following Table contains the result of the calculations that relate to the permanent quantities. In the first and second columns, we have the names of the stars, and their assigned relative distances. The third gives the apparent angular motions, and the fourth their direction. The fifth contains the direction of the same motions, with respect to the parallactic motions arising from the given solar direction; and the sixth gives the velocity of the stars which produce the quantity of the apparent motions.

Table X.

Names of the Stars.	Proportional Distances.	Apparent Motions.	Direction with the Parallel.	Direction with the parallactic Motion.	Velocity of the Stars.
		"			
Sirius -	1,00	1,11528	68.49.40,7 *sp*	10.24.44,3 *sf*	1115281
Arcturus	1,20	2,08718	55.29.49,0 *sp*	0. 0. 8 *sp*	2504621
Capella -	1,25	0,46874	71.35.22,4 *sf*	24.40.21 *sf*	579668
Lyra -	1,30	0,32435	56.20.57,3 *nf*	92.49.30 *nf*	421657
Rigel -	1,35	0,16273	79.29.33,9 *np*	159.28. 1 *np*	219684
α Orionis	1,35	0,13038	85.38.14,6 *nf*	169.18.58 *np*	176010
Procyon	1,40	1,23941	50. 2.24,5 *sp*	9.40.46 *sp*	1735172
Aldebaran	1,40	0,12341	76.29.37,3 *sf*	13.41.48 *sf*	172778
Pollux -	1,42	0,65987	0. 0. 0 *prec.*	61.30.34 *sp*	923528
Spica -	1,44	0,19102	84. 5. 1,8 *np*	144.13.16 *np*	275065
Antares -	1,46	0,26000	90. 0. 0 *north*	178.57.44 *np*	379600
Altair -	1,47	0,71912	48.40.12,0 *nf*	103.17.29 *nf*	1057105
Regulus -	1,48	0,22888	20.27.37,5 *np*	70. 9.20 *sp*	338711
β Leonis	1,50	0,55324	7.16. 8,4 *sp*	40.34.31 *sp*	829856
β Tauri -	1,50	0,10039	84.58.27,1 *sf*	13.17.11 *sf*	150579
Fomalhaut	1,50	0,30698	11,16.16,3 *nf*	16.47. 5 *sf*	460469
α Cygni	1,60	0,06440	27.45.56,3 *np*	177.31.39 *np*	103036
Castor -	2,00	0,13294	17.30.40,6 *sp*	45.25.48 *sp*	265869
α Ophiuchi	2,00	0,07698	40.30.24,8 *sf*	33.29.28 *sf*	153955
α Coronæ	2,00	0,23279	7.24.15,4 *sf*	105. 0.43 *nf*	465587
α Aquarii	2,00	0,20615	67.10.17,1 *np*	162.43.46 *nf*	412295
α Andromedæ	2,00	0,09268	40.20.48,2 *sf*	12.55.11 *sf*	185360
α Serpentis	2,00	0,21913	60. 7.12,5 *nf*	161.34. 4 *nf*	438257
α Pegasi	2,00	0,18917	72. 5.16,0 *np*	157.45.25 *nf*	378388
α Hydræ	2,30	0,16598	57.30.24,8 *np*	107. 6.24 *np*	381763
α' Libræ -	2,40	0,18376	54.42.59,9 *np*	127. 8. 7 *np*	441022
γ Pegasi	2,50	0,17355	59.48. 7,9 *np*	174. 5.15 *nf*	433880
α Arietis -	2,50	0,11587	37. 9.15,9 *sf*	29.32.47 *sf*	289685
α Ceti -	2,80	0,14406	33.44. 2,9 *np*	141.18.55 *np*	403356
α Herculis	3,00	0,29000	90. 0. 0 *north*	168.23.41 *nf*	690000
β Virginis	3,00	0,77706	17.59.25,5 *sf*	111.11.44 *nf*	2331169
γ Aquilæ	3,00	0,19320	55.54.41,7 *np*	178.25.20 *nf*	579589
α² Capricorni	3,50	0,26452	79.23.35,3 *nf*	136.21.18 *nf*	925819
β Aquilæ	4,00	0,35127	85. 7.37,0 *sp*	89.49.15 *sp*	1405079
α' Capricorni	4,20	0,28000	90. 0. 0 *north*	146.59.44 *nf*	1176000
α' Libræ	6,00	0,20898	59.27.58,4 *np*	131.46. 7 *np*	1253875

The contents of this Table will enable us to examine the motions of the stars in different points of view. For instance, by the apparent motions in the third column, and their directions in the fourth, a figure may be drawn which will represent the actual state of the heavens, with respect to those annual changes in the situations of our 36 stars, which in astronomical tables are called their proper motions.

Fig. 1, Plate IV. gives us these motions brought into one view, so that by supposing successively every one of the stars to be represented by the central point of the figure, we may see the angular quantity and direction of the several annual proper motions represented by the line which is drawn from the centre to each star. By this means we have the comparative arrangement and quantity of these movements with respect to their directions.

Fig. 3 represents the same motions, but instead of being drawn so as to show their directions with regard to the several meridians and parallels of the stars, they are laid down by the angles contained in the fifth column; and will therefore indicate their arrangement with respect to a line drawn from the solar apex towards the parallactic centre. These directions will remain the same, whatever may be the velocity of the solar motion upon which we shall ultimately fix, provided no change be made in the situation of the apex towards which the sun has been admitted to move.

In these two figures, the lines drawn from the centre give us only the angular changes of the places that have been either observed or calculated, and not the velocities which are required in the stars to produce them. It will therefore be necessary to represent the velocities by two other figures, in

which the same directions are preserved, but where the extent of each line is made proportional to the distance of the stars in the second column.

Fig. 2 is drawn according to this plan; the angles of the directions remain as in the fourth column, but the lines are lengthened so as to give us the velocities contained in the sixth.

In Fig. 4, the angles of the 3d figure are preserved, but the lines are again lengthened as in Fig. 2.

N. B. These two last figures would have been of an inconvenient size if they had been drawn on the same scale with the two foregoing ones, for which reason, in comparing the 2d and 4th with the 1st and 3d, it must be remembered that the former are reduced to one half of the dimensions of the latter.

Remarks on the sidereal Motions as they are represented from Observation.

As we have now before us a set of figures which give a complete view of the result of the calculations contained in the Xth Table, we may examine the arrangements of the stars, and draw a few conclusions from them, that will throw some light upon the subject of our present inquiry.

In the first place, then, we have to observe in Fig. 1, that 17 out of the 21 stars, whose motions are directed towards the north, are crowded together into a compass of little more than 76½ degrees. But this figure, as we have shown, is drawn from observation. We are consequently obliged to conclude, that, if these motions are the real ones, there must be some physical cause which gives a bias to the directions in

which the stars are moving; if so, it would not be improbable that the sun, being situated among this group of stars, should partake of a motion towards the same part of the heavens.

Our next remark concerns the velocity of the sidereal motions; and therefore we must have recourse to Fig. 2, where we perceive that the greatest motions are not confined to the brightest stars. For instance, the velocity of β Virginis is but little inferior to that of Arcturus, and exceeds the velocity of Procyon. Likewise the velocities of β Aquilæ, α' Libræ, and α' Capricorni, surpass that of Sirius; and an inspection of the rest of the figure will be sufficient to show how very far the velocities of Capella, Lyra, Rigel, α Orionis, Aldebaran, and Spica, are exceeded by those of many other stars.

If we look at the arrangement of the stars with respect to the direction of the solar motion, we find in Fig. 3, that a somewhat different scattering of them has taken place; but still most of the stars appear to be affected by some cause which tends to lead them to the same part of the heavens, towards which the sun is moving; and the directions of the greatest number of them are not very distant from the line of the solar motion.

The whole appearance of this figure presents us with the idea of a great compression above the centre, arising from some general cause, and a still greater expansion in the lower part of it. The considerable projection of a few stars on both sides, is however a plain indication that the compressing or dilating cause does not act in their directions.

When the velocity of the stars, represented in the same point of view in Fig. 4, is examined, we find a particularity

in the direction and comparative velocities in the largest stars that must not be overlooked. Four of them, Rigel, α Orionis, Spica, and Antares, have a motion towards that part of the heavens in which the solar apex is placed, and their motions are very slow. Three other stars of the 1st magnitude, Arcturus, Procyon, and Sirius, move towards the opposite part of the heavens, and their motions, on the contrary, are very quick.

The direction of the motion of Aldebaran, compared with its small velocity, is no less remarkable; and seems to be contrary to what has been pointed out with the three last mentioned stars; we shall however soon have an opportunity of showing that it is perfectly consistent with the principles of the solar motion.

The Solar Motion and its Direction assigned in the first Part of this Paper are confirmed by the Phenomena attending the observed Motions of the 36 Stars.

An application of some of the foregoing remarks will be our next subject; and I believe it will be found, that in the first place they point out the expediency of a solar motion. That next to this they also direct us to the situation of the apex of this motion: and lastly, that they will assist us in finding out the quantity requisite for giving us the most satisfactory explanation of the phenomena of the observed proper motions of the stars.

In examining the second figure, it has been shown that no less than six stars of the first magnitude, namely, Capella, Lyra, Rigel, α Orionis, Aldebaran, and Spica, have less velocity than nine or ten much smaller stars. Aldebaran and

α Orionis indeed have so little motion that there are but three stars in all the 36 that have less. But the situation of these bright stars, from their nearness, must be favourable to our perceiving their real motions if they had any, unless they were counteracted by some general cause that might render them less conspicuous. Now to suppose that the largest stars should really have the smallest motions, is too singular an opinion to be maintained; it follows, therefore, that the apparently small motions of these large stars is owing to some general cause, which renders at least some part of their real motion invisible to us. But when a solar motion is introduced, the parallax arising from that cause will completely account for the singularity of these slow motions.

If the foregoing argument proves the expediency of a solar motion, its direction is no less evidently pointed out by it. For if the parallax occasioned by the motion of the sun is to explain the appearances that have been remarked, it will follow, that a direction in opposition to the motion of Arcturus, will answer that end in the most satisfactory manner. That compression, for instance, which has been remarked in the motions of the stars moving toward the solar apex in Fig. 8, and which is so completely accounted for by a parallactic motion arising from the motion of the sun, points out the direction in which the sun should move, in order to produce this required parallactic motion. The expansion of the motions that are in opposition to the former is evidently owing to the same parallactic motions, which in this direction unite with the real motions of the stars; and as, in the former case, the observed motions are the differences between the parallactic and real motions, so here they are the sum of them.

The remark that stars having a side motion, are not affected by the cause of the compression or expansion, which acts upon the rest, is perfectly explained; for a parallactic motion, in the direction of the motion of Arcturus, can have no effect in lengthening or shortening the perpendicular distance to which a star may move in a side direction.

I have only to add, that the small velocities of Rigel, α Orionis, Spica, and Antares, in Fig. 4, as well as the great velocities of Arcturus, Procyon, and Sirius, point out the same apex which in the first part of this Paper has already been established by more extended computations.

The case of Aldebaran, though seemingly contrary to what has been shown, confirms the same conclusions. This will appear by considering that a star, moving towards the solar apex with a greater real motion than its parallactic one, must continue apparently to move in its real direction; but should a star, such as Aldebaran, move towards the apex with less velocity than the parallactic motion which opposes it, there will arise a change of direction, and the star will be seen moving towards the opposite part of the heavens.

Trial of the Method to obtain the Quantity of the Solar Motion by its Rank among the sidereal Velocities.

According to the conditions that have been explained, a calculation may be made with a view of equalizing the velocities of the sun and the star α Orionis; and the result of it will show that the proposed equality will be obtained when the solar motion is 1″,866880. It will moreover be found than so small an increase of this motion as 0″,01 would give us 19 stars with less, and 17 with more velocity than that

which the calculation assigns to the sun; this consequently fixes one of the limits to which the solar motion ought not to come up, if we intend it should hold a middle rank among the sidereal velocities.

On the other hand, by a similar calculation of the velocities of the star Pollux and the sun, it appears that a solar motion of 0″,967754 will make them equal; and that a diminution of this motion not exceeding 0″,01 would give us 19 stars moving at a greater rate than the sun, and only 17 falling short of its velocity. This consequently fixes the other limit to which the solar motion ought not to be depressed. And thus it appears by this method, that the quantity we are desirous of ascertaining, is confined within very narrow bounds, and that by fixing upon a mean of the two limits, we may have the rank of the solar motion true to less than 0″,15.

Calculations for investigating the Consequences arising from any proposed Quantity of Solar Motion, and for delineating them by proper Figures.

Before we can justly examine the real motions of stars which it will be necessary to admit in consequence of a given solar motion, it will be convenient to have them represented in two figures that we may see their arrangement and extent; and as a calculation of the required particulars will oblige us to fix upon a certain quantity, we shall take the motion that has been ascertained to belong to the middle rank of the sidereal velocities for a pattern. The result of the necessary calculations is as follows.

Table XI.

Names.	Parallactic Motion.	Real Motion.	Parallactic Angle.	Velocity.
	″	″	° ′ ″	
Sun - -	0,00000	1,116992	00.00.00	1116992
Sirius - -	0,75697	0,395212	149.20. 6 *sf*	395212
Arcturus -	0,59847	1,488713	179.59.55.7 *sp*	1786455
Capella -	0,88905	0,506128	22.29.12,5 *nf*	682654
Lyra - -	0,36349	0,498949	40.29.14 *nf*	648634
Rigel - -	0,55470	0,709381	4.36.52 *np*	957665
α Orionis -	0,71410	0,842559	1.38.38 *np*	1137455
Procyon -	0,74161	0,523428	156.32.21 *sp*	732799
Aldebaran -	0,72736	0,608148	2.45.15 *nf*	851407
Pollux - -	0,78643	0,743971	50.12.11 *np*	1056489
Spica - -	0,74009	0,902004	7. 6.44 *np*	1298886
Antares -	0,74110	1,000835	0.16.10,5 *np*	1461219
Altair - -	0,64544	1,071042	40.48. 4 *nf*	1574431
Regulus -	0,75095	0,706833	17.43.53 *np*	1046113
β Leonis -	0,68003	0,443842	54.10.14,5 *np*	665763
β Tauri -	0,73063	0,633317	2. 5.15,5 *nf*	949976
Fomalhaut	0,66693	0,383414	13.22. 5,5 *nf*	575121
α Cygni -	0,46516	0,529508	0.18. 2,2 *np*	847204
Castor - -	0,55841	0,474647	11.30.32 *np*	949293
α Ophiuchi	0,85202	0,990934	8.28.49 *nf*	381869
α Coronæ -	0,23427	0,870580	87.21.17 *nf*	741160
α Aquarii -	0,55743	0,756754	4.38.19,5 *nf*	1513508
α Andromedæ	0,55889	0,464035	2.33.34 *nf*	928071
α Serpentis	0,38655	0,598458	6.38.54 *nf*	1196917
α Pegasi -	0,55567	0,734265	5.35.47,5 *nf*	1468530
α Hydræ -	0,46554	0,588281	17. 8.26 *np*	1238046
α' Libræ -	0,43377	0,563892	15. 4.29 *np*	1353342
γ Pegasi -	0,44540	0,618272	1.39.27 *nf*	1545679
α Arietis -	0,48898	0,342934	9.35.29,5 *nf*	857336
α Ceti -	0,33271	0,454165	11.26. 5,5 *np*	1271662
α Herculis	0,21909	0,446795	5.56.38,5 *nf*	1340388
β Virginis -	0,36039	0,967572	48.29. 2,5 *nf*	2902716
γ Aquilæ -	0,30898	0,502168	0.36.25 *nf*	1506503
α'' Capricorni	0,31390	0,537285	19.51.52,5 *nf*	1880497
β Aquilæ -	0,24870	0,226458	96.36.59,5 *sp*	905830
α' Capricorni	0,26151	0,519230	17. 4.54,5 *nf*	2180769
α' Libræ -	0,17347	0,349371	26.29.44,5 *np*	2096229

By the contents of this Table, Fig. 5 is drawn with the lines contained in the third column and the angles of the fourth; the scale of it is that of the 5th and 3d figures; and it represents the directions and angular quantities of the real motions that are required to compound with the parallactic effects of the second column, so as to produce those annual proper motions which are established by observation.

Fig. 6 is drawn on the reduced scale of the 2d and 4th figures. The lines make the same angles with the direction of the solar motion as before, but their lengths are in the proportion of the velocities contained in the last column.

Remarks that lead to a necessary Examination of the Cause of the sidereal Motions.

The first particular that will strike us when we cast our eye on Fig. 5, is the uncommon arrangement of the stars. It seems to be a most unaccountable circumstance that their real motions should be as represented in that figure; indeed, if we except only ten of the stars, all the rest appear to be actuated by the same influence, and, like faithful companions of the sun, to join in directing their motions towards a similarly situated part of the heavens.

This singularity is too marked not to deserve an examination; for unless a cause for such peculiar directions can be shown to exist, I do not see how we can reconcile them with a certain equal distribution of situations, quantities, and motions, which our present investigation seems to demand. In order to penetrate as far as we can into this intricate subject, we shall take a general view of the causes of the motions of celestial bodies.

A motion of the stars may arise either from their mutual gravitation towards each other, or from an original projectile force impressed upon them. These two causes are known to act on all the bodies belonging to the solar system, and we may therefore reasonably admit them to exert their influence likewise on the stars. But it will not be sufficient to know a general cause for their motions, unless we can show that its influence will tend to make them go towards a certain part of the heavens rather than to any other. Let us examine how these causes are acting in the solar system.

The projectile motions of the planets, the asteroids, and the satellites, excepting those of the Georgium Sidus, are all decidedly in favour of a marked singularity of direction. We may add to them the comet of the year 1682, whose regular periodical return in 1759 has sufficiently proved its permanent connection with the solar system. Here then we have not less than 23 various bodies belonging to the solar system to show that this cause not only can, but in the only case of which we have a complete knowledge, actually does influence the celestial motions, so as to give them a very particular appropriate direction. Even the exception of the Georgian satellites may be brought in confirmation of the same peculiarity; for though they do not unite with the rest of the bodies of our system, they still conform among each other to establish the same tendency of a similar direction in their motion round the primary planet. And thus it is proved that the similar direction of the motion of a group of stars may be ascribed to their similar projectile motions without incurring the censure of improbability.

Let us however pursue the objection a little farther, and as

we have shown that the celestial bodies of the solar system actually have these similar projectile motions, it may be required that we should also prove that the stars have them likewise; since the appearances in Fig. 3 may otherwise be looked upon as merely the consequence of the assumed solar motion. To this I answer, that setting aside the solar motion, and allowing the observations of astronomers on the proper motion of the stars to give us the real direction and angular quantity of these motions, even then the same similarity will equally remain to be accounted for. In my examination of Fig. 1 and 3, it has been shown that we ought to ascribe the similar directions of the sidereal motions to some physical cause, which probably exerts its influence also on the solar motion; therefore in reverting to those figures I may be said to appeal to the actual state of the heavens, for a proof of what has been advanced, with respect to the similarity of the directions of projectile motions.

Having thus examined one cause of the sidereal motions, and shown that as far as we are acquainted with its mode of acting in the solar system, it is favourable to a similarity of direction; and that moreover, if we ascribe the motion of the stars to it, we have also good reason, from observation, to believe it to be in favour of the same similarity; we may in the next place proceed to consider the mutual gravitation of the stars towards each other. This is an acknowledged principle of motion, and the laws of its exertion being perfectly known, we shall in this inquiry meet with no difficulty relating to its direction, which is always towards the attracting body.

Considerations of the attractive Power required for a sufficient Velocity of the sidereal Motions.

As attraction is a power that acts at all distances, we ought to begin by examining whether the motions of our stars can be accounted for by the mutual gravitation of neighbouring stars towards each other, or by a periodical binal revolution of them about a common centre of gravity; or whether we ought not rather to have recourse to some very distant attractive centre. This may be decided by a calculation of the effects arising from the laws according to which the principle of attraction is known to act. For instance, let the sun and Sirius be two equal bodies placed in the most favourable situation to permit a mutual approach by attraction: that is, let them be without projectile motions, and removed from all other stars which might impede their progress towards each other, by opposite attractions. Then, by calculation, the space over which one of them would move in a year, were the matter of both collected in the other as an attractive centre, would be less than a five thousand millionth part of a second; supposing that motion to be seen by an eye at the distance of Sirius, and admitting the parallax of the whole orbit of the earth on this star to be one second.

This proves evidently that the mere attraction of neighbouring stars acting upon each other cannot be the cause of the sidereal motions that have been observed.

In the case of supposed periodical binal revolutions of stars about a common centre of gravity, where consequently projectile motions must be admitted, the united power of the connected stars, provided the mass of either of them did not

greatly exceed that of the sun, would fall very short of the attraction required to give a sufficient velocity to their motions. The star Arcturus for example, which happens to move, as is required, in an opposite direction to the proposed solar motion, were it connected with the sun, and had the proper degree of necessary projectile motion, could not describe an arch of 1″ of its orbit, about their common centre, in less than 108 years; and though the opposite motion of the sun, by a parallactic effect would double that quantity, it still would fall short of the change we observe in this star in the course of a single year.

Other considerations are still more against the admission of such partial connections: they would intirely oppose the similarity of the directions of the sidereal motions that have been proved to exist, and which we are now endeavouring to explain.

Let us then examine in what manner a distant centre of attraction may be the cause of the required motions. By admitting this centre to be at a great distance, we shall have its influence extended over a space that will take in a whole group of stars, and thus the similar directions of their motions will be accounted for. Their velocities also may be ascribed to the energy of the centre, which may be sufficiently great for all the purposes of the required motions. A circumstance, however, attends the directions of the motions to be explained, which shows that a distant centre of attraction alone will not be sufficient; for these motions, as we may see in Fig. 3, though pretty similar in their directions, still are diverging; whereas if they were solely caused by attraction, they would converge towards the attracting centre, and point

out its situation. It is therefore evident that projectile motions must be combined with attraction, and that the motions of the stars when regulated in this manner, are not unlike the disposition by which the bodies of the solar system are governed. If we pursue this arrangement, it will be proper to consider the condition, and probable existence of such a centre of attraction.

There are two ways in which a centre of attraction, so powerful as the present occasion would require, may be constructed; the most simple of them would be a single body of great magnitude; this may exist, though we should not be able to perceive it by any superiority of lustre; for notwithstanding it might have the usual starry brightness, the decrease of its light arising from its great distance would hardly be compensated by the size of its diameter; but to have recourse to an invisible centre, or at least to one that cannot be distinguished from a star, would be intirely hypothetical, and, as such, cannot be admitted in a discussion, the avowed object of which is to prove its existence.

The second way of the construction of a very powerful centre, may be joint attraction of a great number of stars united into one condensed group.

The actual existence of such groups of stars has already been proved by observations made with my large instruments; many of those objects, which were looked upon as nebulous patches, having been completely resolved into stars by my 40 and 20-feet telescopes. For instance, the nebula discovered by Dr. HALLEY in the year 1714, in which the discoverer, and other observers after him, have seen no star,

I have ascertained to be a globular cluster, containing, by a rough calculation, probably not less than fourteen thousand stars. From the known laws of gravitation, we are assured that this cluster must have a very powerful attractive centre of gravity, which may be able to keep many far distant celestial bodies in control.

But the composition of an attractive centre is not limited to one such cluster. An union of many of them will form a still more powerful centre of gravitation, whose influence may extend to a whole region of scattered stars. To prove that I argue intirely from observations, I shall mention that another nebula, discovered by Mr. MESSIER in the year 1781, is, by the same instruments, also proved to consist of stars; and though they are seemingly compressed into a much smaller space, and have also the appearance of smaller stars, we may fairly presume that these circumstances are only indications of a greater distance, and that, being a globular cluster, perfectly resembling the former, the distance being allowed for, it is probably not less rich in the number of its component stars. The distance of these two clusters from each other is less than 12 degrees, and we are certain that somewhere in the line joining these two groups there must be a centre of gravitation, far superior in energy to the single power of attraction that can be lodged in either of the clusters.

I have selected these two remarkable objects merely for their situation, which is very near the line of the direction of the solar motion; but were it necessary to bring farther proof of the existence of combined attractions, the numerous objects

of which I have given catalogues* would amply furnish me with arguments.

If a still more powerful but more diffused exertion of attraction should be required than what may be found in the union of clusters, we have hundreds of thousands of stars, not to say millions, contained in very compressed parts of the milky way, some of which have already been pointed out in a former Paper.† Many of these immense regions may well occasion the sidereal motions we are required to account for; and a similarity in the direction of these motions will want no illustration.

With regard to the situation of the condensed parts of the milky way, and of the two clusters that have been mentioned, we must remark, that the seat of attraction may be in any part of the heavens whatsoever; for when projectile motions are given to bodies that are retained by an attractive centre, they may have any direction, even that at right angles to its situation not excepted.

It will give additional force to the arguments I have used for the admission of far distant centres of attraction, as well as projectile motions in the stars that are connected with them, when we take notice that, independent of the solar motion, and setting that intirely aside, the action of these causes will be equally required to explain the acknowledged proper motions of the stars. For if the sun be at rest, then Arcturus must actually change its place more than 2″ a year, and consequently this and many other stars, which are well known to change their situation, must be supposed to have

* Phil. Trans. for 1786, page 457; for 1789, page 212; for 1802, page 477.

† Ibid. for 1802, page 495.

projectile motions, and to be subject to the attraction of far distant centres.

Determination of the Quantity of the Solar Motion.

If I am not mistaken, it will now be allowed that no objection can arise against any solar velocity we may fix upon, for want of a cause that may be assigned to act upon the sun, and many stars, so as to account for their motions, and similar tendency towards a certain part of the heavens; we may consequently proceed in examining whether the quantity that has been assumed for calculating the contents of the XIth Table, will sufficiently come up to the conditions we have adopted for directing our determination.

In Fig. 6 we have the velocities of the 36 stars delineated, and by examining the last column of the Table from which they are taken, we find that the parallactic effects arising from the proposed solar motion require the velocity of 18 stars to exceed that of the sun, and exactly the same number to be inferior to it; so far then the rank which has been assigned to the solar motion is a perfect medium among the sidereal velocities.

If we examine in the next place how this motion will agree with a mean rate deduced from the velocities in the above mentioned column, we find a 36th part of their sum to be 1.196550. A solar motion, therefore, which agrees with this mean rate will differ from one assigned by the middle rank no more than 0",079558; and, on account of the smallness of this quantity, the calculations required to lessen it, by some little increase of the solar motion, might well be dispensed with; but if we were desirous of greater precision, the

secondary purpose, next to be considered, would rather incline us to an opposite alteration.

The great disparity of the sidereal motions, which has been mentioned as an incongruity in the first part of this Paper, and has more evidently been shown to exist when we examined the representations of these motions in the 3d figure, is the next point we have to consider in the effect of the solar motion. Let us see how far we have been successful in lessening the ratio these velocities bear to each other. The last column of the Xth Table contains them as they must have been admitted if the sun had been at rest. The proportion of the quickest motion to the slowest is there as 2504621 to 103036; and the velocity of one is therefore above 24 times greater than that of the other. But in consequence of the solar motion we have used, the two extreme velocities are reduced to 2902716 and 395212; which gives a proportion of less than $7\frac{1}{2}$ to 1.

If the quantity of the solar motion were lessened to 1″, we might bring the ratio of the extreme velocities so low as 6 to 1; but as the middle rank has already given it a little below the mean rate, I do not think that we ought to lower it still more; so that when all circumstances are properly considered, there is a great probability that the quantity assumed in the last calculation may not be far from the truth. It appears, therefore, that in the present state of our knowledge of the observed proper motions of the stars, we have sufficient reason to fix upon the quantity of the solar motion to be such as by an eye placed at right angles to its direction, and at the distance of Sirius from us, would be seen to describe annually an arch of 1″,116992 of a degree; and its

velocity, till we are acquainted with the real distance of this star, can therefore only be expressed by the proportional number of 1116992.

Concluding Remarks and Inferences.

We have now only to notice a few remarks that may be made, by way of objection to the solar motion I have fixed upon. If the quantity of this motion is to be assigned by the mean rank of sidereal velocities, it may be asked, will not the addition of every star, whose proper motion shall be ascertained, destroy that middle rank, which has been established? To this I shall answer, that future observations may certainly afford us more extensive information on the subject, and even show that the solar motion should not exactly hold that middle rank, which from various motives we have been induced to assign to it; but at present it appears, that according to the doctrine of chances, a middle rank among the sidereal velocities must be the fairest choice, and will remain so, unless, what is now a secondary consideration, should hereafter become of more importance than the first. That this should happen is not impossible, when a general knowledge of the proper motions of all the stars of the 1st, 2d, and 3d magnitudes can be obtained; but then the method of calculation that has been traced out in this and the former Paper, is so perfectly applicable to any new lights observation may throw upon the subject, that a more precise and unobjectionable solar motion can be ascertained by it with great facility. Hitherto we find that a mean rank agrees sufficiently with the phenomena that were to be explained: the apparent velocities of Arcturus and Aldebaran, without a solar motion

for instance, were to each other, in the IXth Table, as 208 to 12; our present solar motion has shown, that when the deception arising from its parallactic effect is removed by calculation, these velocities are to each other only as 179 to 85, or as 2 to 1. And though Arcturus still remains a star that moves with great velocity, yet in the XIth Table we have 4 or 5 stars with nearly as much motion; and 4 with more.

Our solar motion also removes the deception by which the motion of a star of the consequence of α Orionis is so concealed as hardly to show any velocity; whereas by computation we find that it really moves at a rate which is fully equal to the motion of the sun.

I must now observe, that the result of calculations founded upon facts, such as we must admit the proper motions of the stars to be, should give us some useful information, either to satisfy the inquisitive mind, or to lead us on to new discoveries. The establishment of the solar motion answers both these ends. We have already seen that it resolves many difficulties relating to the proper motions of the stars, and reconciles apparent contradictions; but our inquiries should not terminate here. We are now in the possession of many concealed motions, and to bring them still more to light, and to add new ones by future observations should become the constant aim of every astronomer.

This leads me to a subject, which though not new in itself, will henceforth assume a new and promising aspect. An elegant outline of it has long ago been laid before the public in a most valuable paper on general Gravitation, under

the form of "Thoughts" on the subject;* but I believe, from what has been said in this Paper, it will now be found that we are within the reach of a link of the chain which connects the principles of the solar and sidereal motions with those that are the cause of orbitual ones.

A discovery of so many hitherto concealed motions, presents us with an interesting view of the construction of that part of the heavens which is immediately around us. The similarity of the directions of the sidereal motions is a strong indication that the stars, having such motions, as well as the sun, are acted upon by some connecting cause, which can only be attraction; and as it has been proved that attraction will not explain the observed phenomena without the existence of projectile motions, it must be allowed to be a necessary inference, that the motions of the stars we have examined are governed by the same two ruling principles which regulate the orbitual motions of the bodies of the solar system. It will also be admitted that we may justly invert the inference, and from the operation of these causes in our system, conclude that their influence upon the sidereal motions will tend to produce a similar effect; by which means the probable motion of the sun, and of the stars in orbits, becomes a subject that may receive the assistance of arguments supported by observation.

What has been said in a paragraph of a former Paper, where the sun is placed among the insulated stars,† does not contradict the present idea of its making one of a very extensive system. On the contrary, a connection of this na-

* See the note Phil. Trans. for 1783, page 283. † Ibid. for 1802, page 478.

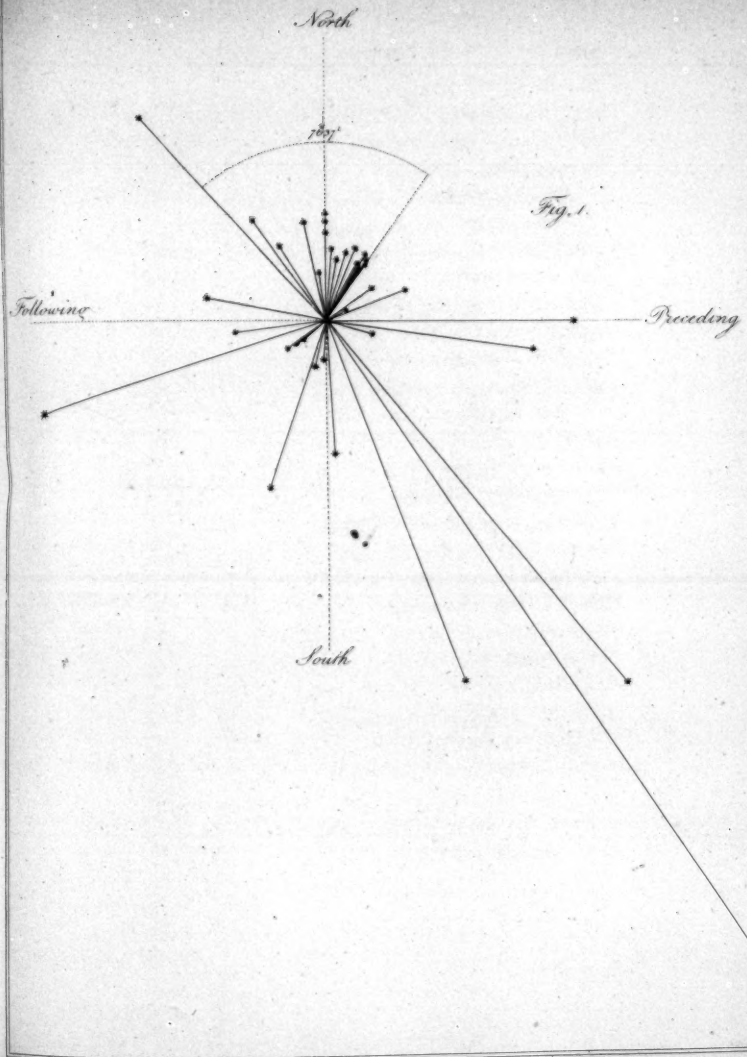
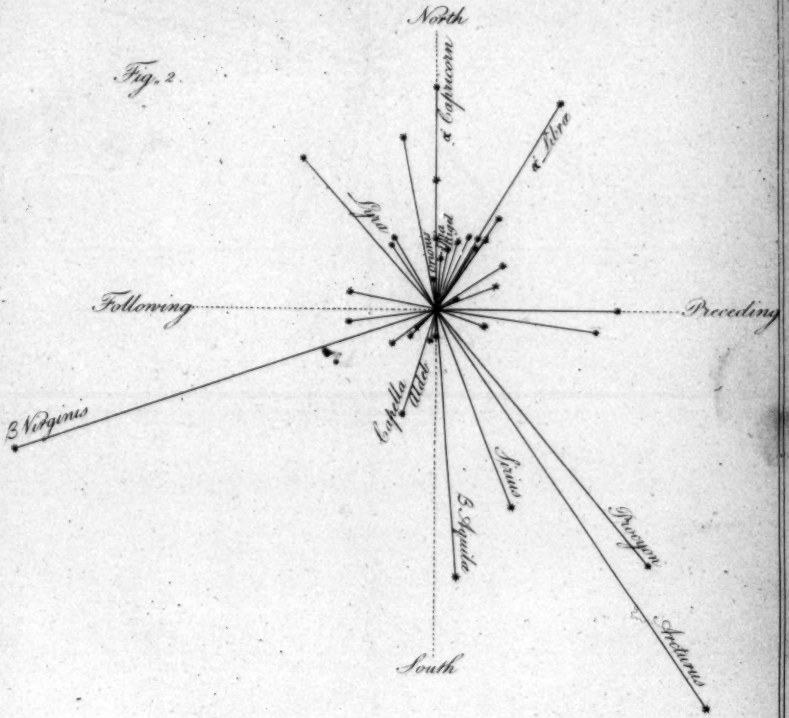


Fig. 1.

Basire sc.

Fig. 2.


North
South
Following
Preceding
Arcturus
Spica
Libra
Regulus
Procyon
Sirius
Capella
Aldeb.
Rigel
Aldebaran
Pleiades
Betelgeux
Orion
Castor
Pollux
Procyon
Sirius
Regulus
Spica
Arcturus
Libra
Aldebaran
Capella
Rigel
Betelgeux
Pleiades
Orion
Castor
Pollux
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Procyon
Sirius
Regulus
Spica
Arcturus
Libra
Aldebaran

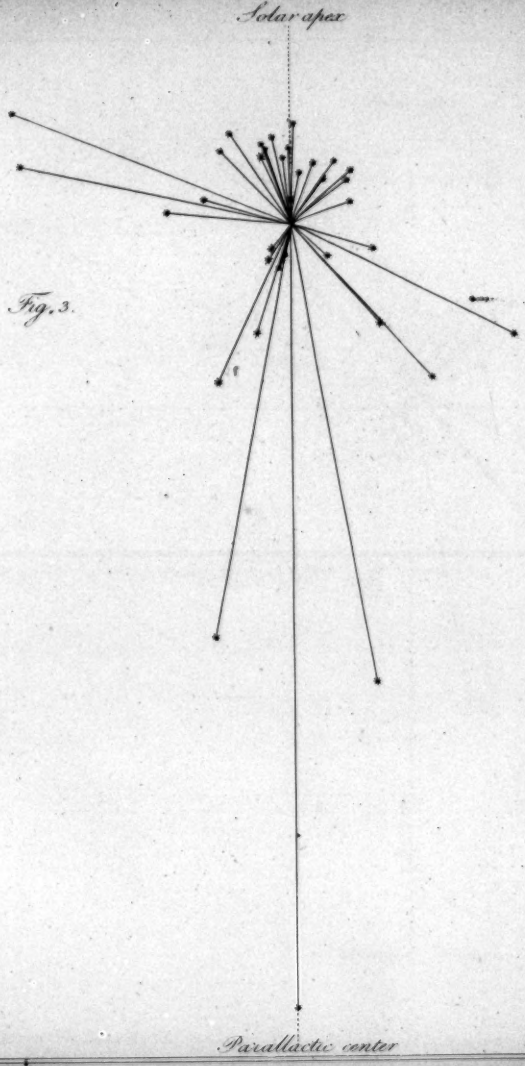
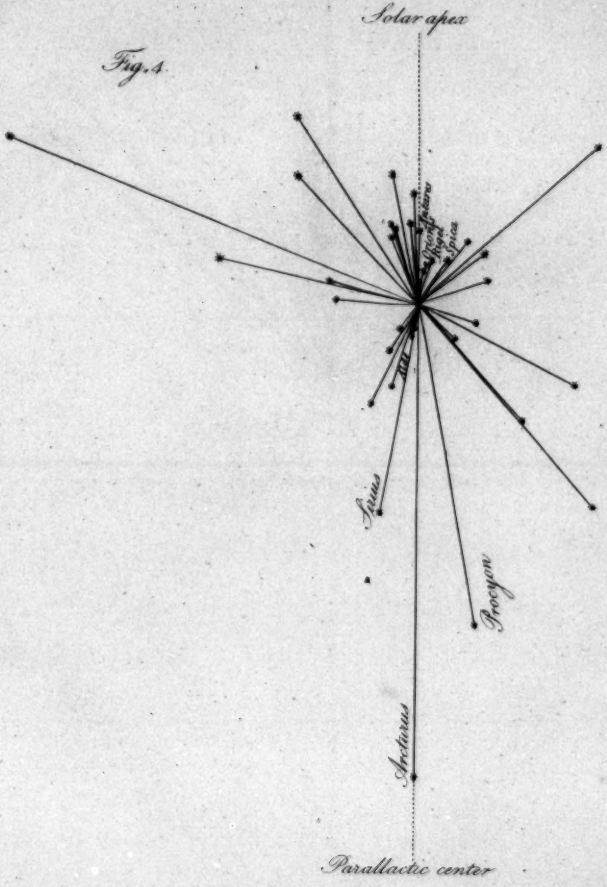


Fig. 3.

Basire sc.

Fig. 4.



Basire fc

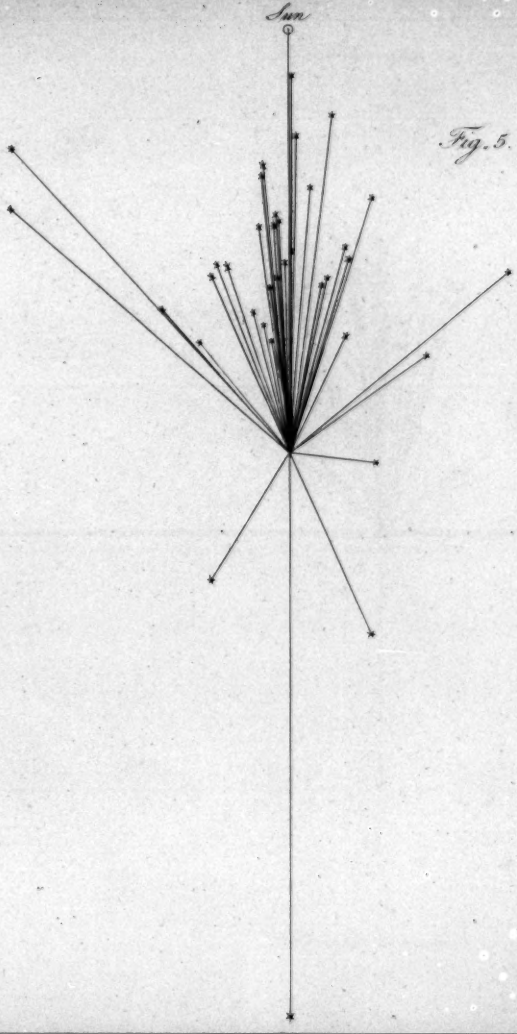
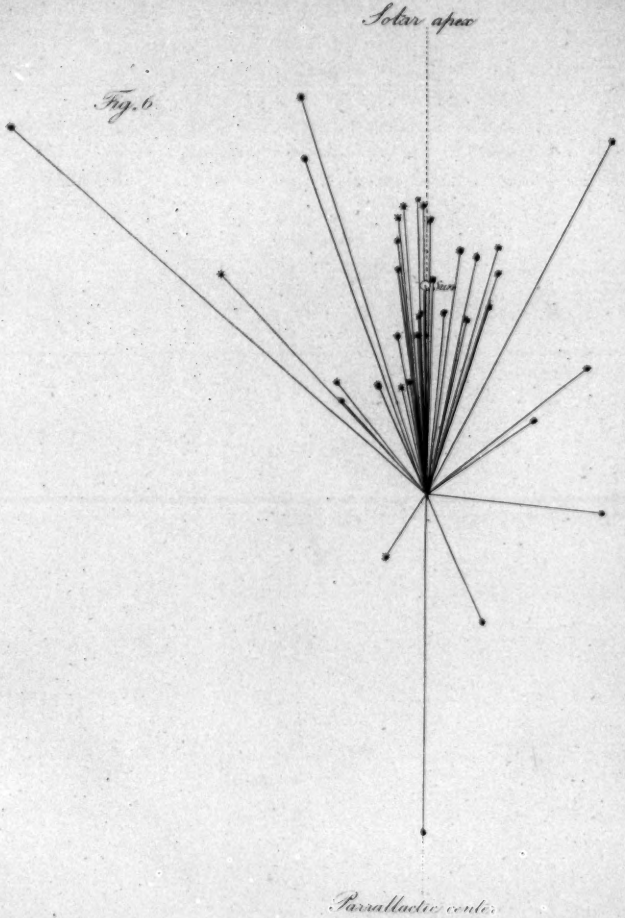


Fig. 5.


Solar apex
Fig. 6.
Parallactic centre.

ture has been alluded to in the same Paper.* The insulation ascribed to the sun relates merely to a supposed binary combination with some neighbouring star; and it has now been proved by an example of Arcturus, that the solar motion cannot be occasioned or accounted for by a periodical revolution of the sun and this or any other star about their common centre of gravity.

* Phil. Trans. for 1802, page 479.

ERRATA

In Table VII. of the first part of this Paper, star Aldebaran, the two last columns,
for 13° 18′ 58″, *read* 13° 41′ 48″.
for 0,02842, *read* 0,02922.

Printed by W. Bulmer and Co.
Cleveland-Row, St. James's.

METEOROLOGICAL JOURNAL,

KEPT AT THE APARTMENTS

OF THE

ROYAL SOCIETY,

BY ORDER OF THE

PRESIDENT AND COUNCIL.

METEOROLOGICAL JOURNAL

for January, 1805.

1805	Six's Therm. least and greatest Heat.	Time. H.	M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrometer.	Rain. Inches.	Wind. Points.	Str.	Weather.
	°			°	°		°				
Jan. 1	29	8	0	29	44	29,85	79		NE	2	Cloudy.
	33	2	0	33	47	29,80	80		ENE	2	Cloudy.
2	33	8	0	37	45	29,75	94	0,030	E	1	Foggy.
	40	2	0	40	47	29,78	95		E	1	Foggy.
3	37	8	0	37	46	30,01	95	0,045	E	1	Foggy.
	41	2	0	41	49	30,05	95		E	1	Cloudy.
4	38	8	0	40	47	30,07	95	0,040	ENE	1	Cloudy.
	43	2	0	43	52	30,07	91		SE	1	Fair.
5	37	8	0	40	48	29,96	93		ESE	1	Cloudy.
	43	2	0	43	49	29,91	93		SE	1	Cloudy.
6	41	8	0	43	49	29,93	94	0,016	SSE	1	Cloudy.
	45	2	0	45	52	30,00	93		SW	1	Cloudy.
7	41	8	0	46	50	29,78	94		S	2	Cloudy.
	48	2	0	48	52	29,84	93		WNW	2	Cloudy.
8	33	8	0	33	49	30,40	90	0,130	WNW	2	Fine.
	42	2	0	42	53	30,41	90		SW	1	Fine.
9	34	8	0	35	50	30,35	92		SW	1	Foggy.
	35	2	0	34	52	30,31	92		SW	1	Foggy.
10	25	8	0	26	48	30,16	90		ENE	1	Cloudy.
	30	2	0	30	51	30,07	92		NE	1	Cloudy.
11	29	8	0	34	49	29,95	92		NE	1	Foggy.
	34	2	0	32	51	29,85	90		E	1	Cloudy.
12	27	8	0	27	47	29,58	87		ENE	1	Cloudy.
	32	2	0	32	49	29,40	88		E	1	Cloudy.
13	32	8	0	39	48	29,09	90		SE	2	Cloudy.
	44	2	0	44	51	28,98	91		SE	2	Cloudy.
14	39	8	0	39	48	28,88	93	0,350	SSW	1	Cloudy.
	44	2	0	44	51	29,04	90		S	1	Cloudy.
15	36	8	0	36	49	29,24	93		SE	1	Fair.
	41	2	0	41	52	29,37	91		S	1	Cloudy.
16	35	8	0	38	48	29,72	93	0,050	S	1	Fair.
	47	2	0	47	51	29,68	90		S	2	Cloudy.

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1805	Sir't Therm. Range and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	A. M.	P. M.	Inches.		Inches.	Points.	Str.	
	°		°	°		°				
Jan. 17	44	8 o	44	50	29.55	95	0,140	SSW	1	Fair.
	48	2 o	48	54	29.55	87		SSW	1	Fair.
18	54	8 o	54	50	29.70	91	0,165	SSW	1	Fine.
	41	2 o	40	53	29.76	87		W&W	1	Cloudy.
19	52	8 o	33	50	29.84	90		S	1	Fine.
	43	2 o	42	52	29.85	89		SSE	2	Cloudy.
20	57	8 o	41	50	28.98	93	0,110	ESE	1	Cloudy.
	45	2 o	45	52	28.89	94		SE	1	Rain.
21	53	8 o	33	50	29.24	94	0,112	WNW	1	Fine.
	40	2 o	40	54	29.30	91		NE	1	Cloudy.
22	52	8 o	34	50	29.30	93	0,033	NE	1	Snow.
	36	2 o	36	52	29.07	91		NE	1	Cloudy.
23	53	8 o	34	50	29.35	93	0,052	NE	1	Cloudy.
	36	2 o	36	52	29.31	93		NE	1	Snow.
24	52	8 o	32	49	29.61	91		ENE	2	Cloudy.
	33	2 o	33	50	29.64	89		NE	2	Cloudy.
25	51	8 o	34	48	29.86	88		NE	2	Cloudy.
	37	2 o	37	49	29.72	86		NE	2	Cloudy.
26	50	8 o	30	47	29.77	84		NE	2	Cloudy.
	31	2 o	30	48	29.77	83		NE	2	Cloudy.
27	49	8 o	30	46	29.73	84		NE	2	Cloudy.
	32	2 o	31	49	29.72	83		NE	2	Cloudy.
28	49	8 o	30	47	29.61	90		ENE	1	Cloudy.
	30	2 o	29	48	29.52	90		E	1	Cloudy.
29	49	8 o	32	46	29.16	92		ENE	1	Snow.
	37	2 o	36	49	29.04	87		ENE	1	Cloudy.
30	54	8 o	35	47	28.90	94	0,200	ENE	1	Cloudy.
	38	2 o	38	49	28.95	94		E	1	Cloudy.
31	52	8 o	35	47	29.21	93	0,060	E	1	Cloudy.
	37	2 o	37	50	29.38	91		NE	1	Cloudy.

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1805	Six's Therm. least and greatest Heat.	Time. H. M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Winds. Str.	Weather.
	°		°	°		°				
Feb. 1	31	7 0	31	46	29,64	88		NE	2	Cloudy.
	33	2 0	32	48	29,70	83		NE	2	Cloudy.
2	23	7 0	23	45	29,85	85		NE	2	Fine.
	36	2 0	34	49	29,88	84		NW	1	Fair.
3	27	7 0	28	45	30,07	84		W	1	Cloudy.
	39	2 0	39	48	30,04	85		W	1	Fair.
4	34	7 0	37	46	29,84	93		W	1	Rain.
	48	2 0	44	49	29,51	94		S	1	Rain.
5	44	7 0	47	48	28,94	97	0,502	S	1	Cloudy.
	48	2 0	45	50	28,86	87		W	2	Cloudy.
6	31	7 0	31	47	29,66	83	0,030	NNE	2	Fine.
	37	2 0	37	49	29,85	76		NW	2	Fine.
7	31	7 0	31	47	29,95	87		W	1	Fair.
	44	2 0	44	50	29,83	83		SSW	2	Cloudy.
8	38	7 0	43	48	29,57	95	0,170	SSW	1	Cloudy.
	51	2 0	50	52	29,58	92		SW	1	Cloudy.
9	48	7 0	50	50	29,76	95	0,092	SSW	2	Cloudy.
	54	2 0	54	53	29,84	90		SW	2	Cloudy.
10	48	7 0	48	52	29,81	94		SW	1	Cloudy.
	53	2 0	53	55	29,72	88		SSW	1	Cloudy.
11	43	7 0	43	53	29,69	92		SSW	1	Cloudy.
	45	2 0	43	54	29,76	91		NE	1	Rain.
12	35	7 0	36	51	29,91	80		NE	2	Cloudy.
	41	2 0	41	55	30,01	81		NNE	2	Fair.
13	34	7 0	35	51	30,21	87		N	1	Cloudy.
	40	2 0	40	54	30,22	80		NW	1	Cloudy.
14	30	7 0	30	51	30,07	85		W	1	Fine.
	41	2 0	41	54	30,03	83		NW	1	Fine.
15	36	7 0	36	52	30,03	84		W	1	Cloudy.
	40	2 0	40	53	30,07	83		N	1	Cloudy.
16	32	7 0	33	50	30,13	80		E	1	Cloudy.
	41	2 0	41	53	30,07	80		E	1	Cloudy.

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1805	Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	A. M.	P. M.	Inches.		Inches	Points	Str.	
	°		°	°		°				
Feb. 17	28	7 0	28	50	29.95	83		ENE	1	Fine.
	39	2 0	39	53	29.88	81		E	1	Fine.
18	32	7 0	33	50	29.82	81		E	1	Cloudy.
	40	2 0	40	54	29.83	78		ENE	1	Fine.
19	29	7 0	29	49	29.90	87		NE	1	Fine.
	41	2 0	41	53	29.96	82		NE	1	Fair.
20	29	7 0	29	49	30.04	88		NW	1	Fine.
	43	2 0	43	53	30.00	86		SW	1	Hazy.
21	29	7 0	39	50	29.90	95	0.085	SSW	1	Cloudy.
	51	2 0	48	54	29.74	95		S	2	Rain.
22	43	7 0	43	53	29.63	91	0.020	SW	1	Cloudy.
	51	2 0	51	55	29.72	76		SW	2	Fair.
23	40	7 0	40	53	29.91	86		WSW	2	Fine.
	50	2 0	50	55	30.07	75		W	2	Fair.
24	45	7 0	47	54	30.00	94	0.048	SSW	1	Cloudy.
	50	2 0	50	56	29.75	92		SSW	2	Rain.
25	39	7 0	39	54	29.78	90	0.098	W	2	Fine.
	50	2 0	50	57	29.85	76		W	2	Fair.
26	45	7 0	46	55	29.78	86		W	2	Cloudy.
	52	2 0	52	58	29.85	77		W	2	Cloudy.
27	43	7 0	43	56	29.70	90		SW	1	Cloudy.
	51	2 0	50	58	29.64	86		SSW	1	Cloudy.
28	46	7 0	46	56	29.50	80		W	2	Cloudy.
	49	2 0	47	58	29.58	76		WNW	2	Cloudy.

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1805	Six's Therm. lower tod greatest Heat.	Time. H.	M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrom.	Rain. Inches.	Wind. Points.	Wind. Str.	Weather.
Mar. 1	38	7	10	40	55	29,80	71		WNW	2	Cloudy.
	46	1	10	44	57	29,52	80		NW	2	Fair.
2	34	7	10	35	54	29,89	88		WNW	1	Fair.
	47	1	10	46	56	29,69	80		WNW	2	Cloudy.
3	35	7	10	37	54	30,14	88		W	1	Cloudy.
	47	1	10	47	56	30,18	77	0,055	SW	1	Fair.
4	44	7	10	W47	54	30,02	93	0,070	SW	1	Rain.
	54	1	10	W53	57	30,02	80		W	2	Cloudy.
5	45	7	10	47	55	29,88	87		WSW	2	Cloudy.
	63	1	10	W53	52	29,80	73		NW	2	Fair.
6	39	7	10	W40	53	30,02	85		W	1	Fair.
	52	1	10	W52	56	30,07	71		NW	2	Fair.
7	38	7	10	38	53	30,18	80		NE	1	Cloudy.
	48	1	10	W43	55	30,04	74		SSE	2	Cloudy.
8	36	7	10	W36	52	30,08	77		E	2	Cloudy.
	44	1	10	W43	56	30,00	73		SE	2	Fair.
9	31	7	10	W31	52	29,77	81		E	1	Fine.
	49	1	10	W39	54	29,66	76		E	1	Hazy.
10	38	7	10	W34	51	29,82	71		NE	1	Cloudy.
	42	1	10	W42	54	29,52	71		NE	1	Fair.
11	30	7	10	W31	51	29,86	81		E	1	Fine.
	46	1	10	W46	53	29,80	74		S	1	Fine.
12	40	7	10	W39	52	29,67	90		S	2	Rain.
	61	1	0	61	55	29,75	73		S	2	Fair.
13	48	7	0	48	54	30,05	87		SSW	2	Fine.
	62	1	0	62	57	30,02	76		S	2	Fine.
14	50	7	0	50	55	29,82	81		S	2	Fine.
	60	1	0	60	58	29,70	75		S	2	Fine.
15	44	7	0	44	57	29,77	85		SSW	1	Fine.
	53	1	0	55	58	29,85	75		NW	2	Cloudy.
16	39	7	0	40	57	29,97	84		WSW	1	Fine.
	54	1	0	55	59	29,97	78		SSW	1	Fair.

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1805.	Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	A. M.	P. M.	Inches.		Inches.	Points.	Str.	
	°		°	°		°				
Mar. 17	43	7	41	56	29.87	85		S	2	Cloudy.
	56	2	50	60	29.72	73		S	2	Fine.
18	47	7	46	58	29.98	97	0.057	SW	1	Rain.
	54	2	51	58	29.97	88		SE	1	Cloudy.
19	49	7	49	57	30.19	84		NNE	1	Cloudy.
	52	2	52	60	30.20	69		NE	2	Fine.
20	41	7	41	57	30.13	90	0.205	NE	1	Rain.
	45	2	45	58	30.13	90		ENE	1	Rain.
21	42	7	42	56	30.15	90	0.225	NE	1	Cloudy.
	48	2	47	59	30.06	78		NE	1	Fair.
22	39	7	39	59	29.96	87		NE	1	Cloudy.
	50	2	48	58	29.96	82		E	1	Fair.
23	39	7	39	56	30.00	84		NE	2	Fine.
	49	2	48	58	30.07	77		ENE	2	Fine.
24	35	7	37	56	30.21	83		ENE	2	Fine.
	47	2	47	57	30.22	73		ENE	2	Fine.
25	31	7	31	53	30.14	84		NE	2	Fine.
	47	2	47	58	30.02	77		NE	1	Fine.
26	32	7	33	55	29.90	85		NE	1	Fine.
	48	2	48	57	29.90	76		E	1	Fine.
27	34	7	39	54	29.92	83		ENE	1	Cloudy.
	43	2	43	56	29.97	80		NE	1	Cloudy.
28	29	7	30	52	30.12	78		NE	1	Fine.
	41	2	40	55	30.15	73		NE	1	Fair.
29	31	7	34	51	30.13	77		NE	1	Cloudy.
	43	2	41	53	30.08	82		S	1	Rain.
30	39	7	40	53	29.93	92	0.270	SW	1	Cloudy.
	57	2	57	54	29.91	91		SW	1	Cloudy.
31	45	7	46	54	29.87	91		SW	1	Fine.
	57	2	56	59	29.81	73		S	1	Cloudy.

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1805	Six's Therm. least and greatest Heat.	Time. H.	M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Str.	Weather.
	°			°	°		°				
April 1	47	7	0	48	54	29.72	88	0,080	SE	1	Rain.
	57	2	0	57	57	29.78	74		WNW	1	Fair.
2	47	7	0	47	55	29.97	83		NNE	1	Fair.
	59	2	0	58	59	29.98	71		N	1	Fair.
3	46	7	0	48	57	29.98	83		W	1	Cloudy.
	57	2	0	57	59	29.96	68		WNW	1	Fair.
4	45	7	0	45	57	29.47	83	0,170	SW	2	Cloudy.
	49	2	0	47	57	29.47	83		W	2	Rain.
5	38	7	0	40	55	29.04	85	0,045	W	2	Fine.
	51	2	0	48	58	29.58	77		NNW	2	Cloudy.
6	40	7	0	41	56	29.67	86		WNW	1	Rain.
	48	2	0	47	57	29.80	79		NNE	1	Cloudy.
7	43	7	0	43	56	30.16	82	0,088	NE	2	Cloudy.
	53	2	0	52	59	30.24	70		NE	2	Fair.
8	37	7	0	39	58	30.38	87		NE	1	Foggy.
	53	2	0	52	58	30.38	75		ENE	1	Fine.
9	38	7	0	40	56	30.38	85		E	1	Fair.
	54	2	0	54	59	30.34	73		E	1	Fine.
10	39	7	0	41	56	30.27	83		E	1	Fair.
	55	2	0	55	58	30.18	76		E	1	Hazy.
11	41	7	0	43	56	30.00	83		E	1	Hazy.
	58	2	0	57	58	29.92	73		E	1	Hazy.
12	42	7	0	44	56	29.81	81		ESE	1	Fine.
	62	2	0	61	59	29.76	65		S	1	Fine.
13	42	7	0	46	57	29.67	74		SE	1	Hazy.
	64	2	0	63	61	29.61	64		S	1	Fine.
14	47	7	0	48	58	29.45	77		E	1	Hazy.
	59	2	0	59	60	29.41	70		E	2	Fair.
15	43	7	0	45	58	29.44	83		NE	1	Fine.
	55	2	0	54	59	29.48	76		NE	2	Cloudy.
16	43	7	0	45	57	29.50	82		NNW	2	Cloudy.
	53	2	0	49	58	29.52	76		NW	2	Cloudy.

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1805	Six'a Therm. least and greatest Heat.	Time. H.	Time. M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Winds. Str.	Weather.
	°			°	°						
Apr. 17	40	7	0	43	56	29,72	85		NNE	2	Cloudy.
	50	2	0	48	57	29,82	81		NE	2	Cloudy.
18	45	7	0	46	57	29,95	85		NNW	1	Cloudy.
	56	2	0	54	58	29,98	68		NNW	1	Cloudy.
19	45	7	0	47	57	30,05	83		NE	1	Fair.
	58	2	0	57	59	30,08	69		NE	1	Fair.
20	38	7	0	43	57	30,20	84		ENE	1	Hazy.
	61	2	0	58	59	30,20	70		NE	1	Hazy.
21	45	7	0	47	58	30,20	81		WSW	1	Fine.
	63	2	0	62	59	30,15	68		NE	1	Fair.
22	43	7	0	44	58	30,11	82		NE	1	Rain.
	54	2	0	53	59	30,08	69		E	1	Fair.
23	36	7	0	40	57	30,11	79		NE	1	Fine.
	52	2	0	51	59	30,04	69		NE	2	Cloudy.
24	36	7	0	38	56	29,68	81		ENE	1	Hazy.
	52	2	0	52	57	29,49	68		E	2	Cloudy.
25	42	7	0	45	56	29,31	87	0,075	E	1	Cloudy.
	55	2	0	52	58	29,37	85		ENE	1	Rain.
26	42	7	0	43	56	29,44	91	0,330	NE	1	Cloudy.
	57	2	0	56	59	29,45	76		NE	1	Fair.
27	44	7	0	44	56	29,65	86		N	1	Cloudy.
	52	2	0	51	57	29,73	75		N	1	Cloudy.
28	37	7	0	38	56	29,76	86	0,090	NE	1	Rain.
	46	2	0	45	56	29,82	77		NE	1	Cloudy.
29	35	7	0	35	55	29,55	88	0,165	E	2	Snow.
	44	2	0	42	56	29,33	92		NE	2	Rain.
30	36	7	0	37	54	29,65	85	0,540	NE	1	Fair.
	50	2	0	50	57	29,70	69		NE	1	Fair.

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for May, 1805.

1805		Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
			H. M.	A. M.	P. M.	Inches.		Inches.	Points.	Str.	
		°		°	°		°				
May	1	37	7 0	39	55	29,67	86		W	1	Fair.
		53	2 0	53	57	29,61	69		W	1	Cloudy.
	2	40	7 0	42	55	29,57	76		NE	1	Cloudy.
		53	2 0	50	57	29,60	75		NE	1	Hazy.
	3	38	7 0	42	55	29,67	87		NE	1	Hazy.
		55	2 0	55	58	29,62	72		E	1	Fair.
	4	40	7 0	42	56	29,77	86		E	1	Hazy.
		57	2 0	57	58	29,82	67		SW	2	Cloudy.
	5	46	7 0	46	56	29,88	83	0,033	W	2	Cloudy.
		57	2 0	55	58	29,90	73		WNW	1	Cloudy.
	6	40	7 0	44	56	30,04	83	0,025	W	1	Fine.
		62	2 0	61	58	30,02	69		SW	1	Cloudy.
	7	48	7 0	50	57	29,95	84		SW	2	Cloudy.
		59	2 0	59	58	29,90	77		SSW	1	Rain.
	8	49	7 0	51	57	29,57	89	0,275	S	1	Rain.
		53	2 0	49	58	29,50	87		N	2	Rain.
	9	40	7 0	42	57	29,65	85	0,185	N	2	Fair.
		53	2 0	53	58	29,72	75		N	2	Fair.
	10	45	7 0	47	57	29,63	88		SW	2	Rain.
		59	2 0	57	59	29,56	74		W	1	Cloudy.
	11	48	7 0	50	57	29,39	83	0,023	SW	2	Cloudy.
		57	2 0	56	60	29,33	75		WSW	2	Rain.
	12	42	7 0	45	57	29,45	83	0,165	WSW	2	Fair.
		57	2 0	54	59	29,61	74		WSW	2	Fair.
	13	42	7 0	46	57	30,01	82		WNW	1	Fine.
		60	2 0	59	59	30,10	68		N	1	Fair.
	14	43	7 0	46	58	30,19	83		NNE	1	Fine.
		58	2 0	58	60	30,13	71		NNE	1	Cloudy.
	15	40	7 0	45	57	29,94	82		NE	1	Hazy.
		58	2 0	55	59	29,83	75		NE	1	Cloudy.
	16	48	7 0	49	58	29,75	82		SW	1	Fine.
		66	2 0	64	60	29,78	66		SW	1	Fair.

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for May, 1805.

1805	Six's Therm. least and greatest Heat.	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.		A. M.	P. M.	Inches.		Inches.	Points.	Str.	
	°			°	°		°				
May 17	51	7	0	52	59	29,90	81		W	1	Cloudy.
	60	2	0	59	60	29,96	76		N	1	Cloudy.
18	50	7	0	52	59	30,04	82		NE	1	Cloudy.
	60	2	0	59	61	30,04	76		NE	1	Cloudy.
19	50	7	0	52	60	29,98	88	0,145	ENE	1	Cloudy.
	62	2	0	62	61	29,98	77		SE	1	Cloudy.
20	50	7	0	52	60	30,06	86		E	1	Cloudy.
	62	2	0	61	61	30,07	78		E	1	Cloudy.
21	49	7	0	51	60	30,12	86		E	1	Cloudy.
	65	2	0	65	62	30,08	77		ENE	1	Fair.
22	50	7	0	53	61	29,88	85		NE	1	Cloudy.
	67	2	0	64	64	29,83	77		NE	1	Fair.
23	45	7	0	48	61	29,83	75		NE	2	Cloudy.
	58	2	0	53	63	29,86	68		NE	2	Fair.
24	41	7	0	47	60	30,05	78		NE	2	Fair.
	59	2	0	57	62	30,04	69		N	1	Fair.
25	49	7	0	52	61	30,02	78		NE	1	Fair.
	65	2	0	63	62	30,02	71		E	1	Cloudy.
26	44	7	0	48	60	30,02	80		E	1	Fine.
	59	2	0	59	61	30,00	71		E	1	Fine.
27	44	7	0	51	60	30,02	83		NE	1	Fine.
	65	2	0	63	61	30,06	71		E	1	Fine.
28	44	7	0	51	60	30 21	83		NE	1	Hazy.
	60	2	0	60	62	30,22	75		E	1	Fine.
29	44	7	0	49	60	30,26	82		E	1	Fine.
	67	2	0	66	61	30,24	65		E	1	Fine.
30	50	7	0	55	60	30,10	78		SW	1	Hazy.
	72	2	0	72	62	30,15	70		NE	1	Fair.
31	47	7	0	52	61	30,24	78		NE	1	Fair.
	63	2	0	62	62	30,30	69		NE	1	Fine.

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for June, 1805.

1805	Six's Therm. least and greatest Heat.	Time. H.	M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Str.	Weather.
June 1	° 45	7	0	° 49	° 61	30,37	° 80		NE	1	Fair.
	60	2	0	58	61	30,37	70		NE	1	Cloudy.
2	41	7	0	47	60	30,36	81		NE	1	Fine.
	60	2	0	60	61	30,32	71		NE	2	Fair.
3	44	7	0	48	60	30,22	81		W	1	Fine.
	71	2	0	71	62	30,08	70		NW	2	Fine.
4	52	7	0	53	61	30,03	83		NE	1	Cloudy.
	60	2	0	58	61	30,04	73		NE	1	Cloudy.
5	47	7	0	50	60	30,05	72		ENE	1	Cloudy.
	58	2	0	56	60	30,02	68		E	1	Cloudy.
6	48	7	0	50	59	29,89	76		S	1	Fine.
	67	2	0	64	60	29,85	67		W	1	Cloudy.
7	47	7	0	48	59	30,05	77		SW	1	Fine.
	69	2	0	67	61	30,07	67		S	1	Fair.
8	51	7	0	53	60	30,10	84		SW	2	Fair.
	71	2	0	69	61	30,08	68		SSW	2	Fair.
9	56	7	0	59	61	29,89	80		E	1	Cloudy.
	74	2	0	70	63	29,84	78		SSE	2	Cloudy.
10	59	7	0	60	62	29,66	90	0,185	SSE	1	Rain.
	66	2	0	64	63	29,54	88		S	2	Cloudy.
11	56	7	0	59	62	29,43	87	0,215	S	2	Cloudy.
	68	2	0	63	63	29,49	80		S	2	Rain.
12	49	7	0	52	61	29,91	82	0,388	S	2	Fine.
	66	2	0	58	62	30,03	78		S	2	Rain.
13	48	7	0	50	62	30,22	88	0,095	S	1	Fine.
	67	2	0	66	62	30,21	73		S	1	Cloudy.
14	55	7	0	55	62	30,08	87	0,075	S	1	Rain.
	56	2	0	54	62	29,90	88		N	1	Rain.
15	49	7	0	51	61	29,82	92	1,190	N	2	Rain.
	62	2	0	61	62	29,96	70		N	1	Fair.
16	49	7	0	52	61	30,08	82		W	1	Fine.
	66	2	0	66	62	30,06	68		W	1	Cloudy.

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for June, 1808.

1805	Six's Therm. least and greatest Heat.	Time. H.	M.	Therm. without. A. M.	P. M.	Therm. within.	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Str.	Weather.
	°			°	°	°		°				
June 17	54	7	10	56	61		30,02	83		W	1	Cloudy.
	68	2	10	67	62		29,97	67		WNW	1	Cloudy.
18	54	7	10	57	61		29,76	86	0,068	WNW	2	Fair.
	68	2	10	68	62		29,77	68		NW	2	Fair.
19	50	7	0	53	61		30,00	80		NW	1	Fair.
	66	2	0	61	62		30,04	75		N	1	Rain.
20	49	7	10	51	62		29,98	83	0,160	NE	1	Cloudy.
	58	2	10	56	62		30,02	76		NE	1	Cloudy.
21	49	7	10	50	60		30,12	85	0,175	NE	1	Cloudy.
	57	2	10	55	60		30,18	77		NE	1	Cloudy.
22	49	7	10	50	59		30,16	80		NE	1	Cloudy.
	60	2	10	58	60		30,17	73		NE	1	Cloudy.
23	50	7	10	52	60		30,18	76		NE	1	Cloudy.
	65	2	10	63	60		30,20	73		NE	1	Cloudy.
24	54	7	10	55	60		30,09	84		W	1	Hazy.
	75	2	10	74	62		29,87	69		NW	2	Fair.
25	56	7	0	49	61		29,70	70		W	1	Cloudy.
	64	2	0	63	62		29,64	81		WNW	1	Rain.
26	42	7	0	49	61		29,87	80	0,085	W	1	Fair.
	66	2	0	65	62		29,91	68		SW	1	Fair.
27	49	7	0	52	61		29,89	77		SSW	2	Fair.
	67	2	0	65	62		29,85	72		SW	2	Cloudy.
28	54	7	0	55	61		29,82	87	0,200	SSW	1	Cloudy.
	66	2	0	60	62		29,82	83		N	1	Rain.
29	49	7	0	52	61		30,01	85	0,478	NE	1	Fine.
	64	2	0	62	62		30,09	70		NE	1	Hazy.
30	50	7	0	52	61		30,20	85		E	1	Fine.
	71	2	0	68	63		30,18	69		W	1	Fair.

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for July, 1805.

1805		Six's Therm. least and greatest Heat.	Time. H.	Time. M.	Therm. without. A. M.	Therm. within. P. M.	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Winds. Str.	Weather.
		°			°	°		°				
July	1	52	7	0	56	62	30,26	81		N	1	Fine.
		71	2	0	69	63	30,27	70		SW	1	Fair.
	2	58	7	0	58	62	30,16	78		SSW	1	Cloudy.
		68	2	0	66	63	30,05	75		SW	1	Cloudy.
	3	60	7	0	58	62	29,93	87		SW	1	Cloudy.
		67	2	0	67	63	29,87	85		SSW	1	Rain.
	4	57	7	0	59	63	29,74	83	0,020	SSW	1	Fair.
		79	2	0	78	60	29,67	69		SSE	2	Cloudy.
	5	62	7	0	62	65	29,58	82		S	2	Cloudy.
		70	2	0	68	65	29,64	73		S	2	Cloudy.
	6	55	7	0	56	65	29,70	85	0,420	SW	2	Fair.
		69	2	0	67	65	29,78	75		SW	2	Rain.
	7	51	7	0	54	64	29,93	80	0,380	WSW	2	Fair.
		70	2	0	67	65	29,96	75		SW	1	Cloudy.
	8	57	7	0	58	64	29,98	88	0,035	SSW	1	Cloudy.
		62	2	0	61	64	29,97	85		SSW	1	Cloudy.
	9	52	7	0	56	64	30,08	85	0,016	NE	1	Fair.
		70	2	0	69	66	30,08	71		NE	1	Fair.
	10	53	7	0	54	64	30,02	85		W	1	Hazy.
		74	2	0	72	65	29,96	70		NW	1	Cloudy.
	11	57	7	0	59	64	30,05	86		NE	1	Cloudy.
		64	2	0	62	65	30,08	80		NE	1	Cloudy.
	12	52	7	0	56	63	30,11	80		NE	1	Cloudy.
		68	2	0	67	64	30,08	70		NE	1	Fair.
	13	57	7	0	58	64	30,04	81		NNE	1	Cloudy.
		68	2	0	68	65	30,03	70		NE	1	Fair.
	14	51	7	0	54	64	30,03	83		NE	1	Cloudy.
		67	2	0	60	65	30,01	77		NE	1	Cloudy.
	15	52	7	0	54	64	30,04	80		NE	1	Cloudy.
		65	2	0	65	64	30,04	78		NE	1	Cloudy.
	16	54	7	0	57	64	30,07	79		NE	1	Fine.
		69	2	0	68	65	30,09	70		NE	1	Fair.

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1805	Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	A. M.	P. M.	Inches.		Inches.	Points.	Str.	
	°		°	°		°				
July 17	54	7 0	55	64	30,17	80		NE	1	Cloudy.
	63	2 0	63	64	30,17	76		NE	1	Cloudy.
18	50	7 0	53	63	30,16	80		NE	1	Cloudy.
	65	2 0	64	64	30,27	73		NE	1	Cloudy.
19	52	7 0	55	63	30,24	80		NE	1	Cloudy.
	66	2 0	65	64	30,10	75		ENE	1	Fair.
20	55	7 0	57	63	30,03	85		WSW	1	Hazy.
	76	2 0	76	65	29,98	75		SW	1	Cloudy.
21	60	7 0	62	64	29,76	93	0,365	S	2	Rain.
	69	2 0	68	65	29,59	91		S	2	Cloudy.
22	55	7 0	57	64	29,78	90	0,190	WSW	1	Cloudy.
	71	2 0	70	65	29,77	70		WSW	1	Cloudy.
23	59	7 0	60	64	29,47	92	0,230	S	2	Rain.
	71	2 0	68	65	29,31	81		S	2	Cloudy.
24	54	7 0	57	64	29,56	86	0,215	W	2	Cloudy.
	68	2 0	67	65	29,70	80		WNW	2	Fair.
25	55	7 0	57	64	29,94	80		WNW	2	Fair.
	67	2 0	65	64	30,06	75		NW	2	Cloudy.
26	52	7 0	53	64	30,14	85		WSW	1	Fine.
	73	2 0	73	65	30,07	70		SW	1	Fair.
27	57	7 0	61	64	29,80	81		SE	1	Cloudy.
	74	2 0	73	66	29,71	73		S	1	Hazy.
28	59	7 0	61	65	29,73	80		S	2	Cloudy.
	70	2 0	69	66	29,72	76		S	2	Fair.
29	57	7 0	60	65	29,76	85		S	2	Cloudy.
	69	2 0	69	66	29,78	71		S	2	Fair.
30	55	7 0	57	65	29,83	87		S	1	Fine.
	69	2 0	69	66	29,84	72		S	2	Fair.
31	54	7 0	56	65	29,81	86		SSE	2	Cloudy.
	70	2 0	67	66	29,72	78		SSW	1	Cloudy.

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1805	Six'h Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	A. M.	P. M.	Inches.		Inches.	Points.	Str.	
	°		°	°		°				
Aug. 1	55	7 0	57	65	29.66	87	0,070	SSW	1	Cloudy.
	70	2 0	69	65	29.63	77		SSW	1	Rain.
2	58	7 0	59	65	29.52	87	0,315	SE	2	Rain.
	69	2 0	67	66	29.49	82		S	2	Cloudy.
3	58	7 0	60	65	29.75	90	1,055	W	2	Cloudy.
	72	2 0	72	66	29.82	72		W	1	Hazy.
4	52	7 0	54	65	29.95	90		SW	1	Hazy.
	73	2 0	71	68	29.94	73		S	2	Cloudy.
5	61	7 0	63	65	29.56	88		SW	2	Cloudy. [Much wind last night.
	73	2 0	73	67	29.72	67		WSW	2	Fair.
6	55	7 0	57	66	29.76	84		SW	2	Fine.
	73	2 0	72	67	29.80	67		WSW	2	Fair.
7	52	7 0	54	66	29.94	83		SW	1	Fine.
	71	2 0	69	67	29.92	76		S	2	Cloudy.
8	57	7 0	58	65	29.90	86	0,325	SE	1	Cloudy.
	73	2 0	73	67	29.92	72		S	1	Cloudy.
9	58	7 0	60	66	29.92	90	0,275	S	2	Cloudy.
	75	2 0	74	67	29.95	78		S	2	Cloudy.
10	61	7 0	63	66	30.08	87		SW	1	Cloudy.
	74	2 0	73	68	30.10	71		W	1	Fair.
11	58	7 0	60	67	30.12	84		NE	1	Cloudy.
	76	2 0	75	69	30.05	72		S	1	Fine.
12	58	7 0	62	68	29.89	80		NE	1	Fair.
	79	2 0	78	71	29.80	71		SSE	1	Fair.
13	61	7 0	61	69	29.84	85		NE	1	Cloudy.
	73	2 0	70	70	29.97	70		WNW	1	Fair.
14	60	7 0	60	68	30.00	87		W	1	Fair.
	70	2 0	70	69	30.02	70		W	1	Fair.
15	51	7 0	53	65	30.11	84	0,112	W	1	Fine.
	72	2 0	71	68	30.07	67		WNW	1	Cloudy.
16	57	7 0	58	66	30.00	83	0,020	W	1	Cloudy.
	73	2 0	71	67	29.96	72		W	1	Cloudy.

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1805	Six's Therm. least and greatest Heat.	Time. H.	M.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds. Points.	Str.	Weather.
	°			°	°	Inches.	°	Inches.			
Aug. 17	60	7	0	60	67	30,00	90	0,068	NE	1	Cloudy.
	72	2	0	71	68	30,05	72		NE	1	Fine.
18	60	7	0	61	67	29,97	88		W	1	Cloudy.
	73	2	0	71	68	29,88	74		S	2	Cloudy.
19	56	7	0	57	66	29,63	90	0,740	WNW	1	Fair.
	68	2	0	63	67	29,59	80		NW	1	Rain.
20	57	7	0	57	66	29,60	92	0,565	NW	1	Rain.
	68	2	0	68	67	29,70	81		NE	1	Fair.
21	59	7	0	59	65	29,94	91	0,095	NNE	1	Cloudy.
	62	2	0	61	66	30,05	83		NNE	1	Cloudy.
22	57	7	0	58	65	30,20	83		NNW	1	Cloudy.
	68	2	0	66	66	30,22	77		WNW	1	Cloudy.
23	61	7	0	61	66	30,24	92		W	1	Cloudy.
	73	2	0	73	68	30,27	73		WNW	1	Fair.
24	62	7	0	62	66	30,24	85		W	1	Cloudy.
	73	2	0	73	68	30,24	72		NW	1	Fair.
25	58	7	0	59	65	30,12	91		W	1	Cloudy.
	75	2	0	74	68	30,04	73		WSW	1	Fair.
26	59	7	0	61	67	29,98	88		SW	1	Cloudy.
	71	2	0	68	67	29,93	81		SW	1	Cloudy.
27	57	7	0	59	66	29,93	88		SW	2	Fair.
	74	2	0	73	68	29,94	72		W	2	Fair.
28	59	7	0	59	66	29,96	87		SW	2	Cloudy.
	71	2	0	70	67	29,55	78		SW	2	Cloudy.
29	62	7	0	63	66	29,98	88		SW	1	Fair.
	76	2	0	75	69	30,05	75		SW	1	Fair.
30	61	7	0	63	67	30,10	84		NE	1	Cloudy.
	73	2	0	73	69	29,95	78		S	2	Fair.
31	59	7	0	61	67	29,80	87		SW	1	Cloudy.
	71	2	0	69	68	29,70	78		SSW	2	Cloudy.

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1805	Six's Therm. least and greatest Heat.	Time. H.	Time. M.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds. Points.	Winds. Str.	Weather.
	°			°	°	Inches.	°	Inches.			
Sept. 1	58	7	0	59	66	29,63	88		SW	1	Fair.
	72	2	0	72	68	29,67	73		WNW	2	Fair.
2	57	7	0	58	67	29,93	86	0,135	NE	1	Cloudy.
	67	2	0	66	67	29,96	74		NW	1	Cloudy.
3	57	7	0	58	67	29,85	86		SW	1	Cloudy.
	70	2	0	69	67	29,82	76		SW	2	Cloudy.
4	58	7	0	59	66	29,73	90	0,060	SW	1	Cloudy.
	72	2	0	72	67	29,70	75		SSW	1	Fair.
5	61	7	0	62	66	29,60	87	0,022	S	2	Cloudy.
	73	2	0	73	68	29,68	76		S	2	Fair.
6	58	7	0	60	67	29,60	93	0,702	S	1	Rain. [Lightning and thunder.
	71	2	0	70	68	29,48	74		S	2	Cloudy. [At times wind very high.
7	60	7	0	60	66	29,58	83	0,110	S	2	Cloudy.
	67	2	0	66	66	29,33	80		S	2	Cloudy.
8	57	7	0	58	65	29,52	88	0,075	SW	2	Cloudy.
	68	2	0	67	66	29,63	73		SW	2	Fair.
9	52	7	0	53	65	29,93	88		SW	1	Fine.
	69	2	0	69	66	30,01	74		WSW	1	Fair.
10	52	7	0	53	64	30,16	84		SW	1	Fair.
	71	2	0	70	67	30,12	67		SW	1	Fine.
11	62	7	0	62	66	30,06	93		SW	1	Cloudy.
	73	2	0	71	67	30,05	79		SW	1	Cloudy.
12	61	7	0	61	67	30,04	92		S	1	Cloudy.
	72	2	0	70	68	30,04	79		NW	1	Cloudy.
13	53	7	0	54	66	30,10	87	0,015	WSW	1	Fair.
	70	2	0	69	66	30,09	76		W	1	Cloudy.
14	56	7	0	57	66	30,16	88		W	1	Fine.
	73	2	0	72	68	30,18	74		SW	1	Fine.
15	54	7	0	55	66	30,22	87		W	1	Fine.
	72	2	0	70	68	30,15	81		S	1	Fine.
16	56	7	0	56	67	30,02	88		S	1	Hazy.
	75	2	0	75	69	29,93	77		S	2	Fine.

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1805	Six'a Therm. least, and greatest Heat.	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer.	Rain. Inches.	Winds. Points.	Winds. Str.	Weather.
Sep. 17	57	7	0	60	68	29,96	90		S	1	Cloudy.
	72	2	0	72	69	29,94	78		S	2	Fair.
18	58	7	0	59	68	30,00	91		S	1	Foggy.
	79	2	0	79	71	29,95	76		SE	1	Fine.
19	64	7	0	65	69	29,83	85		E	1	Fine.
	71	2	0	71	70	29,86	84		SE	1	Fair.
20	57	7	0	57	69	29,98	83	0,360	WSW	2	Fair. Last night there was much lightning & thunder.
	68	2	0	67	69	30,15	68		WNW	2	Fair.
21	57	7	0	57	67	29,92	83		ESE	2	Cloudy.
	66	2	0	63	68	29,80	78		S	2	Cloudy.
22	52	7	0	55	66	29,89	88		NE	1	Fair.
	65	2	0	64	67	29,98	83		N	1	Rain.
23	50	7	0	52	65	30,12	86	0,046	WNW	1	Cloudy.
	64	2	0	63	65	30,22	80		N	1	Cloudy.
24	46	7	0	48	63	30,14	86		SW	1	Cloudy.
	60	2	0	58	65	30,14	73		NE	1	Fair.
25	43	7	0	43	62	30,18	83		W	1	Fine.
	62	2	0	61	64	30,18	73		WNW	1	Fine.
26	53	7	0	53	62	30,16	87		W	1	Cloudy.
	64	2	0	63	63	30,16	78		NNE	1	Cloudy.
27	50	7	0	50	62	30,24	86		NE	1	Fine.
	65	2	0	64	63	30,24	80		NE	1	Fine.
28	52	7	0	54	62	30,36	90		SW	1	Cloudy.
	64	2	0	64	63	30,41	81		NE	1	Fair.
29	49	7	0	50	62	30,60	90		NE	1	Cloudy.
	62	2	0	61	63	30,61	73		NE	1	Fair.
30	46	7	0	48	61	30,56	88		NE	1	Fine.
	61	2	0	61	62	30,50	80		NE	1	Cloudy.

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1805		Six's Therm. least and greatest Heat.	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
			H.	M.			Inches.		Inches.	Points.	Str.	
		°			°	°		°				
Oct.	1	54	7	0	55	61	30,44	83		NE	2	Cloudy.
		61	2	0	61	61	30,38	78		NE	1	Cloudy.
	2	55	7	0	56	61	30,30	89		NE	1	Cloudy.
		63	2	0	62	63	30,28	77		ENE	1	Fair.
	3	52	7	0	52	62	30,23	87		E	1	Fine.
		61	2	0	61	64	30,20	76		E	1	Fine.
	4	49	7	0	50	61	30,25	88		ENE	2	Fine.
		60	2	0	60	63	30,25	75		E	2	Fine.
	5	46	7	0	48	61	30,31	86		NE	1	Fine.
		60	2	0	60	63	30,31	73		E	1	Fine.
	6	43	7	0	43	61	30,31	85		NE	1	Fine.
		60	2	0	59	61	30,31	78		NE	1	Fine.
	7	39	7	0	41	59	30,34	85		ENE	1	Fine.
		58	2	0	58	62	30,28	78		WNW	1	Fine.
	8	48	7	0	51	60	30,10	90		W	1	Cloudy.
		63	2	0	63	62	30,05	87		WNW	1	Cloudy.
	9	53	7	0	53	61	29,96	92		ENE	1	Cloudy.
		58	2	0	58	62	29,85	82		E	1	Cloudy.
	10	53	7	0	53	60	29,45	93	0,123	E	1	Rain.
		54	2	0	54	62	29,54	84		NE	1	Cloudy.
	11	39	7	0	41	58	29,98	82	0,180	NE	2	Cloudy.
		52	2	0	51	60	30,08	77		NE	1	Cloudy.
	12	36	7	0	36	57	30,09	83		SW	1	Fair.
		52	2	0	52	59	29,95	79		S	1	Fair.
	13	47	7	0	48	57	29,65	83		SE	2	Cloudy.
		53	2	0	53	58	29,60	89		SW	1	Rain.
	14	41	7	0	42	56	29,60	90	0,125	SW	1	Fair.
		56	2	0	56	59	29,54	84		SW	1	Fair.
	15	47	7	0	47	57	29,32	92	0,380	NE	1	Rain.
		54	2	0	53	60	29,43	80		NE	1	Fair.
	16	44	7	0	45	57	29,36	87	0,040	NE	1	Cloudy.
		52	2	0	52	58	29,19	83		NE	2	Cloudy.

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	°			°	°	Inches.	°	Inches.			
Oct. 17	40	7	0	41	55	29,42	83		WNW	2	Cloudy.
	48	2	0	47	57	29,55	85		WNW	1	Cloudy.
18	39	7	0	40	55	29,85	87	0,035	NE	1	Fine.
	51	2	0	51	57	29,95	78		NE	2	Fair.
19	38	7	0	38	55	30,15	86		NE	1	Fine.
	51	2	0	51	58	30,20	80		NE	1	Fair.
20	38	7	0	38	55	30,26	88		NE	1	Foggy.
	52	2	0	51	56	30,22	87		ENE	1	Fair.
21	45	7	0	45	55	30,18	88		NE	1	Cloudy.
	53	2	0	53	57	30,14	83		ENE	1	Cloudy.
22	49	7	0	49	55	30,04	83		E	2	Cloudy.
	52	2	0	52	58	29,93	74		E	2	Fair.
23	42	7	0	42	54	29,77	84		E	2	Fine.
	52	2	0	52	58	29,72	82		E	2	Fine.
24	45	7	0	48	55	29,66	90		E	1	Rain.
	53	2	0	53	57	29,55	90		E	1	Cloudy.
25	52	7	0	52	56	29,35	95	0,265	E	1	Cloudy.
	59	2	0	59	58	29,45	93		S	1	Cloudy.
26	51	7	0	51	57	29,46	95		E	1	Cloudy.
	54	2	0	54	58	29,34	93		E	1	Cloudy.
27	50	7	0	50	57	29,34	97	0,235	NE	1	Rain.
	51	2	0	51	58	29,43	92		NE	1	Cloudy.
28	45	7	0	45	57	29,62	83		NE	2	Cloudy.
	47	2	0	47	58	29,66	76		NE	2	Cloudy.
29	43	7	0	43	56	29,55	90		NE	2	Cloudy.
	47	2	0	47	57	29,58	83		NE	1	Cloudy.
30	37	7	0	37	54	29,84	90		NE	2	Fine.
	48	2	0	47	57	30,05	80		NE	2	Fine.
31	35	7	0	35	54	30,40	86		SW	1	Fair.
	45	2	0	45	55	30,45	86		SSW	1	Foggy.

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1805	Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.			Inches.		Inches.	Points.	Str.	
	°					°				
Nov. 1	37	7 0	37	54	30,36	83		ESE	1	Fine.
	44	2 0	44	54	30,18	71		E	1	Cloudy.
2	30	7 0	37	52	30,07	83		NE	1	Fair.
	48	2 0	48	55	30,10	78		NE	1	Fair.
3	34	7 0	34	52	30,23	86		NE	1	Fine.
	49	2 0	49	54	30,27	82		E	1	Fine.
4	37	7 0	38	52	30,34	90		E	1	Fine.
	49	2 0	49	55	30,34	84		E	1	Fine.
5	31	7 0	35	52	30,41	88				Foggy.
	40	2 0	40	54	30,38	87				Foggy.
6	38	7 0	41	52	30,38	90		SE	1	Cloudy.
	45	2 0	45	54	30,38	88		S	1	Cloudy.
7	44	7 0	45	52	30,38	93		S	1	Cloudy.
	46	2 0	46	54	30,36	94		SSE	1	Cloudy.
8	39	7 0	40	53	30,34	93		SE	1	Foggy.
	41	2 0	41	54	30,34	92		E	1	Cloudy.
9	39	7 0	42	52	30,20	92		E	1	Cloudy.
	42	2 0	42	53	30,21	91		E	1	Cloudy.
10	39	7 0	40	52	30,15	90		E	1	Cloudy.
	44	2 0	44	53	30,15	90		E	1	Cloudy.
11	41	7 0	41	52	30,28	92		E	1	Cloudy.
	44	2 0	42	53	30,34	87		E	1	Cloudy.
12	42	7 0	40	51	30,35	90		E	2	Cloudy.
	43	2 0	43	52	30,28	89		ESE	1	Cloudy.
13	38	7 0	40	51	30,35	93	0,455	ENE	1	Cloudy.
	46	2 0	45	52	30,42	82		NE	1	Cloudy.
14	35	7 0	38	50	30,52	90		NE	1	Cloudy.
	46	2 0	46	52	30,54	83		NE	1	Fair.
15	40	7 0	44	52	30,03	94		NE	1	Cloudy.
	49	2 0	48	53	30,68	80		NE	1	Cloudy.
16	37	7 0	40	52	30,68	90		NE	1	Cloudy.
	47	2 0	47	54	30,68	80		NE	1	Cloudy.

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1805	Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	°	°	Inches.		Inches.	Points.	Str.	
	°					°				
Nov. 17	36	7 0	36	51	30,59	91		ENE	1	Fine.
	44	2 0	44	54	30,50	80		E	1	Fine.
18	33	7 0	33	50	30,28	83		E	1	Fine.
	42	2 0	42	53	30,18	79		E	1	Fair.
19	31	7 0	33	50	30,08	87		W	1	Cloudy.
	41	2 0	38	52	30,05	88		SW	1	Cloudy.
20	36	7 0	40	50	30,20	93		ENE	1	Cloudy.
	45	2 0	44	53	30,28	90		N	1	Fair.
21	32	7 0	32	49	30,46	85		WSW	1	Fine.
	39	2 0	39	51	30,41	81		SW	1	Cloudy.
22	33	7 0	37	48	30,23	90		SW	1	Cloudy.
	47	2 0	47	52	30,16	85		N	1	Cloudy.
23	37	7 0	37	49	30,30	89		N	1	Fair.
	43	2 0	43	53	30,30	83		SW	1	Cloudy.
24	32	7 0	34	49	30,28	88		SW	1	Cloudy.
	44	2 0	43	52	30,24	90		W	1	Cloudy.
25	42	7 0	42	50	30,22	90		WSW	1	Cloudy.
	47	2 0	47	53	30,17	85		WSW	1	Cloudy.
26	42	7 0	42	50	30,22	90		SW	1	Cloudy.
	47	2 0	47	53	30,21	81		WSW	1	Cloudy.
27	42	7 0	42	51	30,20	90		E	1	Cloudy.
	46	2 0	46	53	30,14	83		E	1	Cloudy.
28	37	7 0	38	51	30,01	87		E	1	Cloudy.
	50	2 0	45	53	29,90	87		E	1	Cloudy.
29	48	7 0	44	52	29,55	94		S	1	Cloudy.
	54	2 0	54	55	29,50	93		S	1	Cloudy.
30	53	7 0	54	53	29,28	97	0,340	SSW	2	Cloudy.
	55	2 0	54	57	29,21	94		S	2	Rain.

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1805	Six's Therm. least and greatest Heat.	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
		H. M.	°	°	Inches.	°	Inches.	Points.	Str.	
Dec. 1	43	8 0	43	55	29,12	90	0,060	SW	1	Cloudy.
	46	2 0	46	56	29,15	87		S	1	Cloudy.
2	34	8 0	34	53	29,60	84	0,062	NNW	2	Cloudy.
	38	2 0	38	55	29,76	81		NNW	2	Fair.
3	32	8 0	37	52	30,02	90		W	2	Cloudy.
	44	2 0	43	54	29,94	87		SW	1	Rain.
4	43	8 0	46	53	30,00	96	0,175	SW	1	Cloudy.
	52	2 0	52	55	30,03	96		SW	1	Cloudy.
5	43	8 0	43	54	30,25	91		W	1	Fine.
	49	2 0	49	56	30,27	86		W	1	Fine.
6	43	8 0	46	54	30,22	96		SW	1	Fair.
	53	2 0	52	57	30,14	94		SW	1	Cloudy.
7	49	8 0	51	55	29,93	94		S	2	Cloudy.
	53	2 0	53	57	29,87	94		SW	2	Cloudy.
8	43	8 0	44	54	29,81	90	0,016	SW	1	Cloudy.
	48	2 0	48	57	29,77	83		SW	2	Cloudy.
9	43	8 0	47	54	29,38	90		S	2	Cloudy.
	46	2 0	43	56	29,17	91		NW	2	Rain.
10	35	8 0	38	53	29,15	92	0,725	WSW	1	Rain.
	41	2 0	41	56	29,25	90		NW	1	Fair.
11	31	8 0	32	52	29,44	77		NNW	1	Fine.
	39	2 0	39	54	29,56	85		NNW	1	Cloudy.
12	32	8 0	35	52	29,25	92		NNE	1	Cloudy.
	37	2 0	37	52	29,46	88		N	1	Cloudy.
13	24	8 0	24	48	29,69	80		NNW	1	Fine.
	30	2 0	29	50	29,64	80		NW	2	Fine.
14	29	8 0	30	47	29,62	83		NW	2	Fine.
	36	2 0	34	48	29,64	87		NW	1	Fair.
15	32	8 0	34	46	29,83	91		NW	1	Snow.
	37	2 0	37	48	29,83	90		NW	2	Cloudy.
16	28	8 0	28	45	30,08	85		NNE	2	Fine.
	32	2 0	32	48	30,16	81		NNE	2	Fine.

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1805	Six'd Therm. least and greatest Heat.	Time. H.	M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Rain. Inches.	Winds. Points.	Str.	Weather.
Dec. 17	25	8	0	26	44	30,38	92		NE	3	Fair.
	32	2	0	32	47	30,41	88		NE	1	Cloudy.
18	29	8	0	33	44	30,33	90		W	1	Cloudy.
	39	2	0	38	47	30,29	91		W	1	Fair.
19	38	8	0	35	44	30,07	89		SW	1	Cloudy.
	44	2	0	44	47	29,94	91		S	1	Rain.
20	43	8	0	47	47	29,63	96	0,025	SSW	1	Cloudy.
	52	2	0	52	49	29,48	96		SSW	1	Rain.
21	50	8	0	51	50	29,22	95		SW	2	Cloudy.
	53	2	0	50	53	29,00	93		SSW	2	Rain.
22	47	8	0	42	50	28,85	95	0,375	S	1	Fair.
	46	2	0	46	53	28,81	90		S	1	Fair.
23	35	8	0	36	49	29,00	88		SW	1	Cloudy.
	40	2	0	40	52	29,12	87		S	1	Fair.
24	30	8	0	32	50	29,52	94		N	1	Cloudy.
	40	2	0	39	52	29,70	91		NE	1	Fair.
25	32	8	0	34	47	29,83	90		SW	1	Cloudy.
	46	2	0	43	51	29,68	90		S	1	Rain.
26	42	8	0	45	48	28,96	95	0,310	S	2	Rain. [Much wind last night.
	48	2	0	48	52	28,87	96		S	2	Cloudy.
27	37	8	0	37	48	29,60	90		NE	2	Fine.
	49	2	0	40	52	29,83	88		NE	2	Fine.
28	31	8	0	35	49	29,98	92		SSE	1	Fair.
	42	2	0	42	51	29,81	93		S	2	Rain.
29	42	8	0	46	50	29,87	97	0,035	SSW	1	Cloudy.
	53	2	0	53	52	29,81	97		SW	1	Cloudy.
30	51	8	0	51	52	29,87	96	0,016	SW	2	Rain.
	54	2	0	53	53	29,90	96		SW	1	Cloudy.
31	53	8	0	53	53	29,94	97		S	2	Cloudy.
	54	2	0	54	55	29,82	96		S	2	Cloudy.

1805.	Six's Therm. without.			Thermometer without.			Thermometer within.			Barometer.*			Hygrometer.			Rain.
	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	
	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Inches.	Inches.	Inches.	Deg.	Deg.	Deg.	Inches.
January	48	25	36,2	48	26	36,9	54	44	49,6	30,41	28,88	29,61	95	79	90,6	1,513
February	54	23	40,7	54	23	40,8	58	45	51,8	30,22	28,86	29,82	95	75	85,9	1,045
March	62	29	44,0	62	30	44,3	60	51	55,5	30,24	29,40	29,96	93	69	80,8	0,882
April	64	35	48,0	63	35	48,2	61	54	57,2	30,38	29,31	29,82	92	64	78,7	1,583
May	72	37	55,4	64	39	55,4	64	55	59,2	30,26	29,33	29,90	89	65	77,2	0,851
June	75	41	57,8	74	47	57,8	63	59	61,2	30,37	29,43	29,99	92	67	77,7	3,324
July	79	50	62,1	78	53	62,5	66	62	64,3	30,27	29,31	29,92	92	69	79,6	2,271
August	79	51	65,0	78	53	65,0	71	65	66,8	30,27	29,49	29,92	92	67	80,6	3,535
September	79	45	62,0	79	43	61,8	71	65	66,0	30,61	29,53	30,00	93	67	81,9	1,525
October	63	35	49,6	63	35	49,8	64	54	58,3	30,45	29,19	29,89	97	73	84,9	1,383
November	56	31	41,8	54	32	42,1	57	48	52,2	30,68	29,21	30,24	97	71	87,5	0,795
December	54	25	40,8	54	26	41,4	57	44	51,2	30,41	28,81	29,69	97	60	90,3	1,799
Whole year			50,0			50,5			57,8			29,90			83,0	20,396

* The quicksilver in the bason of the barometer, is 81 feet above the level of low water spring tides at Somerset-house.

Variation of the Magnetic Needle.

1805.

June	-	-	-	-	24° 7′,8 W.
Dip of the needle			-	-	70 21.

Note. Subtract 2′ from the variation of last year, for the error of the instrument.